HYPERBOLIC 3-MANIFOLDS AND THEIR CONSTRUCTIONS

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ABSTRACT. Hyperbolic geometry is one of the three main geometries also including Euclidean and elliptical geometries. In this paper we focus on hyperbolic 3-manifolds and explore the way we can construct them using convex polyhedra.

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1. INTRODUCTION

There are three main types of geometries: Euclidean, elliptical, and hyperbolic. Euclidean geometry as the name suggests was formalized with 5 postulates or axioms in Euclid's famous Elements [ET93] which are:

(1) A straight line can be drawn between any two points.

(2) A finite straight line can be extended into a straight line.

(3) A circle can be drawn with any center and any radius.

(4) All right angles are equal.

(5) (parallel postulate) If a straight line falls on two straight lines in such a manner that the interior angles on the same side are together less than two right angles, then the straight lines, if produced indefinitely, meet on that side on which are the angles less than the two right angles.

The first four postulates are simple and but the fifth seems unnecessarily complicated and this is exactly what mathematicians thought from the moment Euclid published the

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Elements. It was not until 2000 years later that the independence of the fifth postulate was shown by Eugenio Beltrami in 1868.

This denial of the parallel postulate is what ultimately led to the formation of non-Euclidean geometry. Carl Friedrich Gauss was the first to stumble upon this. After trying to prove the parallel postulate for 20 years, he found that changing it led to strange geometries he called *non-Euclidean geometry.*

He specifically considered when the sum of the angles of a triangle is less than π and after several years of investigating this new geometry, Gauss was convinced that there was no inconsistencies. However he did not publish his results but this was later rediscovered in the 1830s by Nikolai Ivanovich Lobachevsky [LP10] and János Bolyai who independently formalized this theory so it was named "Bolyai-Lobachevskian geometry" which we now call *hyperbolic geometry.*

Then in 1854, Bernhard Riemann gave a famous lecture where he created "Riemannian Geometry" and discussed an infinite family of non-Euclidean geometries using his new ideas like Riemannian metrics and Manifolds. The simplest of these non-Euclidean geometries is *elliptical geometry.*

In this paper, we first start by going more into depth about these geometries and discuss the duality between spherical and hyperbolic geometry. Then we formalize hyperbolic space and discussing the conformal ball model and projective disk model. With the formal definition we can talk about a special type of manifold called (X, G)-manifolds which is what a hyperbolic 3-manifold is. Finally, we finish the paper by proving that we can glue sides of convex polyhedra to form these (X, G)-manifolds.

2. Spherical and Hyperbolic Geometry

There are two main non-Euclidean geometries: spherical and hyperbolic. (More generally, the two main geometries should be *elliptical* and hyperbolic but we don't consider the difference between spherical and elliptical space in this paper). One of the very beautiful things is that there is a duality between spherical and hyperbolic geometry. For example consider the parallel postulate: in spherical geometry, we have

Through a point outside a given line, there is no line parallel to the given line.

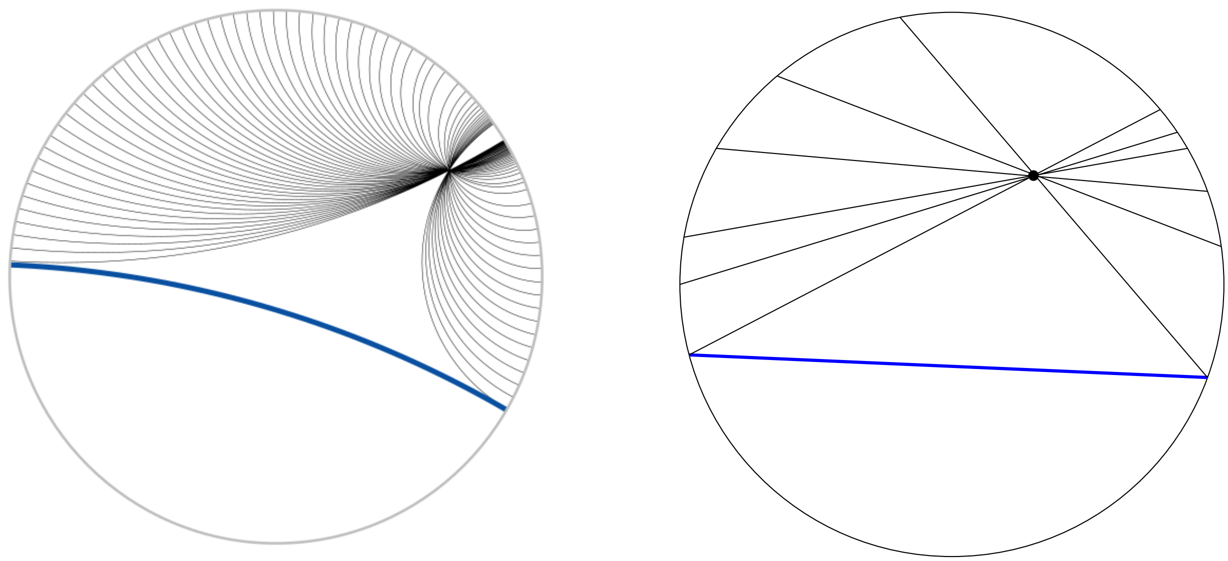
but in hyperbolic geometry, we have

Through a point outside a given line, there are infinitely many lines parallel to the given line.

so hyperbolic geometry is essentially the opposite of spherical geometry.

Additionally, the sum of the angles of a triangle in spherical geometry is always greater than π but a hyperbolic triangle has a sum less than π. However, the sum for a Euclidean triangle is exactly π so we can some what think of Euclidean geometry as the middle ground between spherical and hyperbolic. We can apply this duality to understand more about hyperbolic geometry.

We know that the curvature of a hypersphere of radius r is $1/r^2$ which is always positive so this means that a hyperbolic hyperplane has constant negative curvature. The simplest object with this property is the saddle so we can think of the hyperbolic hyperplane as lots



(a) Parallel lines in the Poincaré Disk model

(b) Parallel lines in the Beltrami-Klein model

Figure 1. Models of Hyperbolic space

of saddles glued together. However this is not the easiest object to visualize and it only gets worst if we increase the number of dimensions. This is why mathematicians have made lots models that map the hyperbolic plane to something easier to visualize [1].

First, there is the *Beltrami-Klein model* named after Eugenio Beltrami and Fleix Klein. Here we talk about the 2-dimensional version but everything can be scaled up to higher dimensions. This model maps the hyperbolic plane onto a disk D in the Euclidean plane. The boundary ∂D is called the *circle at infinity* since these points are not on the hyperbolic plane but the represent the imaginary "points" infinitely far away. They are called *ideal points* and the points outside the disk are called *ultra ideal points*.

Lines in the hyperbolic plane are represented by chords in D so if L is chord, we can easily see that there are infinitely many chords through an outside point P that don't intersect L. See Figure 1b. Now one advantage of this model is straight lines map to straight lines but the Euclidean angles are not the same as the hyperbolic angles so the model is not *conformal*.

There is also the *Poincaré disk model* named after Henri Poincaré. To keep the angles invariant, we can use this model. This model is similar to the Beltrami-Klein model in that it maps the hyperbolic plain to a disk in the Euclidean plane and we use the same terminology. However, lines in the hyperbolic plane map to arcs of circles that are orthogonal to ∂D as shown in Figure 1a. Like the Beltrami-Klein Model, we can see in this figure that there is an infinite number of lines parallel to a line. As mentioned before, the advantage of this model is that it is conformal but a disadvantage is that distances are greatly distorted.

3. Hyperbolic n-space

In this section, we formally define hyperbolic n-space, H^n. Before we go into this, we must first define Euclidean and spherical space.

[1] Lots of these models can be found at [Rat06] and [Thu80].

Definition 3.1. Euclidean n-space denoted with E^n is an inner product space of $\mathbb{R}^n$ with inner product $\cdot$ such that

$$x \cdot y = x_1y_1 + \cdots x_ny_n$$

where $x, y \in \mathbb{R}$.

Definition 3.2. Spherical n-space is

$$S^n = \{x \in \mathbb{R}^{n+1} : |x| = 1\}$$

where $|x| = \sqrt{x \cdot x}$.

Now there are lots of ways we can define hyperbolic space. We can call it the space that has constant negative curvature, the space the follows the modified parallel postulate, etc. We can also define it through models of it but we can see in the previous two models that something is always distorted. The problem is that these models map $H^n \to E^n$ and distortion was proven to be inevitable when mapping to Euclidean space by David Hilbert's theorem in [Hil33]. Therefore, to get around this we must work outside of E^n.

Definition 3.3. Let $x, y \in \mathbb{R}$. The *Lorentizan inner product* is $\circ$ such that

$$x \circ y = x_1y_1 + x_2y_2 + \cdots - x_ny_n.$$

Now $\mathbb{R}^n$ equipped with this inner product is known as *Lorentizan n-space* which is denoted by $\mathbb{R}^{n-1,1}$.

We can define a norm:

Definition 3.4. The *Lorentizan norm* is

$$||x|| = \sqrt{x \circ x}.$$

This norm can be zero, positive, or imaginary. When the $||x||^2 = 0$, we have

$$x \circ x = x_1^2 + x_2^2 + \cdots - x_n^2 = 0$$

so x is on the a $(n-1)$ dimensional cone so if $||x||^2 > 0$, then x is outside the cone and x is inside the cone when $||x||^2 < 0$.

Going back to the duality between spherical and hyperbolic space, we know that the hypersphere with radius r in $\mathbb{R}^{n+1}$ has a constant curvature of $1/r^2$. Since hyperbolic and spherical space are opposites, hyperbolic space should have negative curvature and the only way to make $1/r^2$ negative is if r is imaginary. We cannot usually have an imaginary radius in Euclidean space but we can in Lorentzian space.

Definition 3.5. Hyperbolic n-space is

$$H^n = \{x \in \mathbb{R}^{n+1} : x_{n+1} > 0 \text{ and } ||x||^2 = -1\}.$$

When $||x||^2 = -1$, we have

$$x_1^2 + x_2^2 + \cdots - x_{n+1}^2 = -1$$

so when $x_{n+1} > 0$, then x is on the positive sheet of a $(n-1)$ dimensional hyperboloid of two sheets. This is why this model is named the *hyperboloid model*.

With this, we can formally talk about the Beltrami-Klein Model and the Poincaré Disk Model. We start with the Poincaré Disk model also known as the *conformal ball model*.

3.1. **Conformal Ball Model.** First we define the unit ball

$$B^n = \{x \in E^n : |x| < 1\}.$$

Now we embed this ball in $\mathbb{R}^{n+1}$ by identifying $\mathbb{R}^n$ as $\mathbb{R}^n \times \{0\}$. Next, consider a point $x \in B^n$ and the ray from $-e_{n+1}$ through x. This ray will pass through one and only one point in H^n that we call $\pi(x)$ where $\pi : B^n \to H^n$ is a *stereographic projection*.

Since $\pi(x)$ is on the line in the direction $x + e_{n+1}$, we have

$$\pi(x) = x + s(x + e_{n+1})$$

for some scalar s. Writing this out explicitly gives us

$$\pi(x) = (x_1 + sx_1, \cdots, x_n + sx_n, s)$$

so we can use the condition $||\pi(x)||^2 = -1$ to get

$$x_1^2(1+s)^2 + \cdots + x_n^2(1+s)^2 - s^2 = -1$$

so

$$(1+s)^2|x|^2 - s^2 = -1 \implies (s+1)^2|x|^2 = s^2 - 1 = (s+1)(s-1).$$

Since s must be positive, we can divide both sides by $s+1$ giving us

$$s = \frac{1+|x|^2}{1-|x|^2}.$$

Explicitly,

$$\pi(x) = \left(\frac{2x_1}{1-|x|^2}, \cdots, \frac{2x_n}{1-|x|^2}, \frac{1+|x|^2}{1-|x|^2}\right)$$

which has an inverse of $\pi^{-1} : H^n \to B^n$ where

$$\pi^{-1}(y) = \left(\frac{y_1}{1+y_{n+1}}, \cdots, \frac{y_n}{1+y_{n+1}}\right).$$

This is the map we use to model hyperbolic space.

Now from this projection, we can clearly see that lines in H^n will be curves in B^n and that angles will be invarient. Additionally, notice that when $x \in \partial B^n$, the ray is an asymptote of the hyperboloid which is why we consider these points at "infinity."

3.2. **Projective Disk Model.** The Beltrami-Klein also has another name: the *projective disk model*. Like before define the space we are projecting from: the unit disk

$$D^n = \{x \in \mathbb{R}^n : |x| < 1\}.$$

Note that D^n and B^n contain the same points but mathematicians create a distinction because they define different metrics on D^n and B^n. We shall keep the change in notation to highlight the difference of the two models.

Again we can embed D^n in $\mathbb{R}^{n+1}$ by identifying $\mathbb{R}^n$ as $\mathbb{R}^n \times \{0\}$. Now in this model, we consider a *gnomonic projection* $\mu : D^n \to H^n$. If we have a point $x \in D^n$, we vertically translate it until $x_{n+1} = 1$ and then we radially project it onto H^n. Therefore we have,

$$\mu(x) = \frac{x + e_{n+1}}{|||x + e_{n+1}|||}$$

so the inverse $\mu^{-1} : H^n \to D^n$ is

$$\mu^{-1}(x) = \left(\frac{x_1}{x_{n+1}}, \cdots, \frac{x_n}{x_{n+1}}\right).$$

Let us prove that hyperbolic lines are represented by chords in D^n or more generally, an m-plane in H^n is represented by an m-chord in D^n.

Definition 3.6. An m-plane in H^n is the intersection of of H^n with a $(m+1)$ dimensional vector subspace of $\mathbb{R}^{n+1}$ made of vectors with imaginary Lorentizan norms.

Theorem 3.7. *A subset $P \subseteq D^n$ has the property that $\mu(P)$ is a hyperbolic m-plane if and only if P is the nonempty intersection of an m-plane of $\mathbb{R}^n$ and D^n.*

Proof. Let Q be an m-plane of H^n. This means that Q is the intersection of H^n and a $(m+1)$-dimensional vector subspace V of $\mathbb{R}^{n+1}$ made of vectors with imaginary norms. Now notice that μ^{-1} is first a radial projection onto the hyperplane L through e_{n+1} and then a vertical translation of $-e_{n+1}$. The radial projection maps Q onto $V \cap L$ but since V only contains x such that $||x||^2 < 0$, we know that Q maps onto

$$(U \cap C^n) \cap L = U \cap (L \cap C^n) = U \cap (D^n + e_{n+1})$$

where $U \supseteq V$ is an $(m+1)$-plane in $\mathbb{R}^{n+1}$ and C^n is the n dimensional cone $\{x \in \mathbb{R}^{n+1} : ||x|| = 0\}$. Therefore, when we translate the intersection down, we see that $\mu^{-1}(A)$ is a nonempty intersection of an m-plane of $\mathbb{R}^n$ and D^n. This process can easily be reversed to convert P into a hyperbolic m-plane. ■

Corollary 3.8. *Lines in H^2 are represented by open chords of D^2.*

Proof. A line in H^2 is just a 1-plane in H^2 so μ^{-1} of the line, by Theorem 3.7, is the intersection of a 1-plane of $\mathbb{R}^n$ and D^n which is a chord. ■

4. (X, G)-Manifolds

In this section, we define (X, G)-manifolds and discuss how to create them with convex polyhedra.

Definition 4.1. An n-manifold is a Hausdorff space M such that for each point $x \in M$, there exists an open neighborhood U of x such that U is homeomorphic to an open set in E^n.

Example. Let us prove that a circle, $S^1 = \{x \in \mathbb{R}^2 : |x| = 1\}$, is a 1-manifold. First we define the topology on S^1 as the subspace topology of $\mathbb{R}^2$. Now we can see that S^1 must be Hausdorff since $\mathbb{R}^2$ is Hausdorff.

Let $x \in S^1$ such that $x_2 > 0$. For these points, we consider the neighborhood $U = \{x \in S^1 : x_2 > 0\}$ which we can easily see is open. Now we apply the projection map, which is a homeomorphism, $\pi : \mathbb{R}^2 \to \mathbb{R}$ defined by $\pi(x) = x_1$ to U to get

$$\pi(U) = (-1, 1)$$

which is open in $\mathbb{R}$. We can essentially do the opposite for points where $x_2 < 0$.

The only two points we are missing are $(1, 0)$ and $(-1, 0)$. We cover these points with open neighborhoods $U = \{x \in S^1 : x_1 > 0\}$ and $V = \{x \in S^1 : x_1 < 0\}$, respectively. Now we apply the projection map $\pi' : \mathbb{R}^2 \to \mathbb{R}$ defined by $\pi'(x) = x_2$ to U and V to get open sets in $\mathbb{R}$.

Now we can make give manifolds more structure by making open sets in the manifold homeomorphic to a general geometric space.

Definition 4.2. For a metric space X, a *geodesic arc* is a distance preserving function $\gamma : [a, b] \to X$. That is,

$$d_1(x, y) = d_2(\gamma(x), \gamma(y))$$

for all $x, y \in [a, b]$ where d_1 and d_2 are metrics of $\mathbb{R}$ and X, respectively.

A *geodesic line* is a locally distance preserving function $\lambda : \mathbb{R} \to X$. That is, for each point $a \in \mathbb{R}$, there is an $r > 0$ such that $x, y \in B_r(a)$ implies that

$$d_1(x, y) = d_2(\gamma(x), \gamma(y))$$

where d_1 and d_2 are metrics of $\mathbb{R}$ and X, respectively.

Example. Let us construct a geodesic segment from x to y in E^n which is just the image of a geodesic arc starting from x and ending at y. The general form for a line from x to y is the image of the function $\gamma : [0, |y - x|] \to E^n$ such that

$$\gamma(t) = x + t(y - x).$$

Let t_1 and t_2 be two real numbers in $[0, |y - x|]$ so in order to make γ distance preserving,

$$|\gamma(t_1) - \gamma(t_2)| = |t_1 - t_2|.$$

Simplifying the right hand side gives us

$$|\gamma(t_1) - \gamma(t_2)| = |x + t_1(y - x) - x - t_2(y - x)| = |t_1 - t_2||y - x| = |t_1 - t_2|.$$

From this, we see that $t = s/|y - x|$ for some other parameter s such that our geodesic arc is

$$\gamma(s) = x + s\left(\frac{y - x}{|y - x|}\right).$$

Definition 4.3. An n-dimensional *geometric space* is a metric space X satisfying the following axioms:

(1) There exists a geodesic segment between any two points in X.

(2) Every geodesic arc $\gamma : [a, b] \to X$ can be extended into a geodesic line $\lambda : \mathbb{R} \to X$.

(3) There exists a continuous function $\varepsilon : E^n \to X$ and real $r > 0$ such that ε maps the open ball $B_r(0)$ homeomorphically to $B_r(\varepsilon(0))$.

(4) X is homogeneous.

Notice the parallelism between Euclid's postulates and these axioms.

Example. Euclidean n-space is an n dimensional geometric space. So is S^n with $\varepsilon(0) = e_{n+1}$ and

$$\varepsilon(x) = (\cos|x|)e_{n+1} + (\sin|x|)\frac{x}{|x|} \text{ for } x \neq 0.$$

The same is true for H^n where $\varepsilon(0) = e_{n+1}$ and

$$\varepsilon(x) = (\cosh|x|)e_{n+1} + (\sinh|x|)\frac{x}{|x|} \text{ for } x \neq 0.$$

Now we are ready to talk about (X, G)-manifolds. First let X be a geometric space, let G be a group of similarities of X (which are essentially isometries up to scaling), and let M be an n-manifold. The idea is that we cover M with open connected subsets called *coordinate neighborhoods*. Next we define homemorphisms, called *charts*, between these coordinate neighborhoods and open sets in X and we call the set of all charts an (X, G)-*atlas* for M. These charts need to follow one property: If two coordinate neighborhoods U_i and U_j with charts ϕ_i and ϕ_j, respectively, overlap, then the function

$$\phi_j \circ \phi_i^{-1} : \phi_i(U_i \cap U_j) \to \phi_j(U_i \cap U_j)$$

agrees in a neighborhood of each point of its domain with an element of G.

It turns out that for every (X, G)-atlas for M, there exists a unique maximal (X, G)-atlas for M containing the original atlas (a proof of which we will not cover). We call a maximal (X, G)-atlas for M an (X, G)-*structure* for M.

Definition 4.4. An (X, G)-*manifold* M is an n-manifold M equipped with (X, G)-structure.

Essentially, an (X, G)-manifold is a manifold with some instructions on how its subsets are like subsets of X.

Example. If $\mathrm{I}(X)$ is the group of isometries for a metric space X, a *Euclidean n-manifold* is a $(E^n, \mathrm{I}(E^n))$-manifold, a *spherical n-manifold* is $(S^n, \mathrm{I}(S^n))$-manifold, and a *hyperbolic n-manifold* is a $(H^n, \mathrm{I}(H^n))$-manifold.

5. Gluing Convex Polyhedra

In this section, we cover how we can glue convex polyhedra in $X = E^n$, S^n, or H^n to create Euclidean, spherical, or hyperbolic 3-manifolds. For this section, let X be one of these three spaces.

Definition 5.1. A subset $C \subseteq X$ is called *convex* if for each pair of points $x, y \in C$ such that x and y are distinct and not antipodal when $X = S^n$ there exists a geodesic segment between x and y contained in C.

Definition 5.2. The *dimension* of a convex set C is the least integer m such that C is contained in an m-plane of X. This m-plane is denoted by by $\langle C \rangle$.

Definition 5.3. A *side* of a convex set P is a nonempty, maximal, convex subset of ∂P. If P is nonempty, closed (on some topology) and for each $x \in X$, there is an open neighborhood of x intersecting a finite number of sides of P (or P is *locally finite*), we call P a *convex polyhedron*.

Definition 5.4 (k-face)**.** Let P be an m dimensional convex polyhedron in X. The only m-face of P is P itselt. Suppose that all $(k+1)$-faces of P are already defined. Then the k-face of P is a side of a $(k+1)$-face.

A *proper face* of P is a k-face of P where $k < m$.

Example. As we can see in Figure 2, the square and circle are both convex subsets in E^2. However, each point on the circle is a side so for any $x \in E^2$, there exists no open neighborhood of x that meets only a finite number of sides of the circle. Therefore the circle is not a convex polyhedron but a square is since it clearly has 4 sides.

From here on, when we say "polyhedra" we mean "convex polyhedra." Also we will only be considering $n = 3$ from here on. First we define angles:

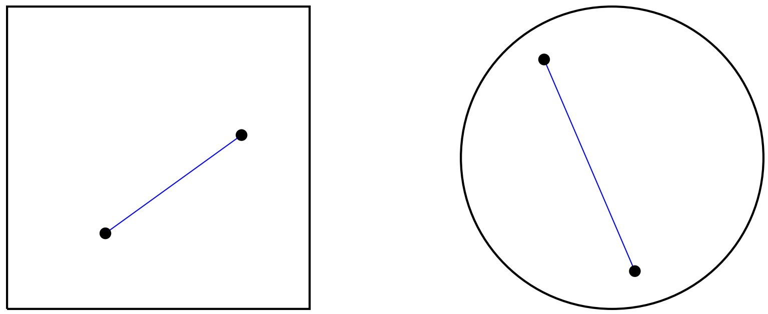


Figure 2. A geodesics segment in a square and circle in E^2

Definition 5.5. Let P be a polyhedron in X and let $x \in P$. The *solid angle* subtended by P at x, is

$$\omega(P, x) = 4\pi \frac{\mathrm{Vol}(P \cap B_r(x))}{\mathrm{Vol}(B_r(x))}$$

where r is less than the distance from x to some side not containing P.

We will not be formally defining volume in this paper but we can think of a solid angle as first calculating what fraction of a small enough ball is inside the polyhedron and then multiplying by 4π.

Let $\mathcal{P}$ be a finite collection of disjoint convex polyhedra in X and let G be a group of isometries of X.

Definition 5.6. A *G-side-pairing* for $\mathcal{P}$ is a subset of G indexed by the set of all sides $\mathcal{S}$ of $\mathcal{P}$

$$\Phi = \{g_S : S \in \mathcal{S}\}$$

such that

(1) there is a side $S' \in \mathcal{S}$ such that $g_S(S') = S$,

(2) the isometries g_S and $g_{S'}$ have the property that $g_{S'} = g_S^{-1}$, and

(3) if S is a side of $P \in \mathcal{P}$ and S' is a side of $P' \in \mathcal{P}$, then
$$P \cap g_S(P') = S.$$

In other words, there is a gluing map assigned for each side in our collection of polyhedra and property (1) tells us that there is a this map glues some other side to this side and we say that these two sides are *paired*. Property (2) makes sure that if two sides are glued together, their gluing maps are inverses of each other. Finally, property (3) tells us that if a gluing map glues sides of two different polyhedra, the gluing map does not glue together any other part of the polyhedra. These properties make sure that gluing polyhedra is like intuitive gluing.

Definition 5.7. Let Φ be a G-side-pairing and let $\Pi = \bigcup_{P \in \mathcal{P}} P$. Two points x and x' in Π are said to be *paired*, notated by $\simeq$, if and only if there is a side S containing x, and x' is in S', and $g_S(x') = x$.

Two points x and y in Π are said to be *related*, notated by $\sim$, if and only if $x = y$ or there is a sequence $x_1, x_2, ...x_m$ such that

$$x = x_1 \simeq x_2 \simeq \cdots \simeq x_m = y.$$

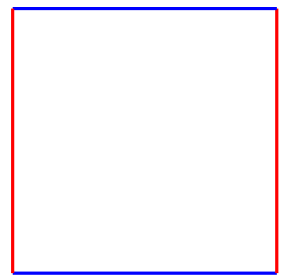


Figure 3. Gluing Pattern for a Torus

Proposition 5.8. *The relation $\sim$ is a equivalence relation*

Proof. First, we know that $a \sim a$ since $a = a$ so $\sim$ is reflexive. Next if $a \sim b$, then there exists a sequence $a_1, a_2, ..., a_n$ such that $a = a_1 \simeq a_2 \simeq \cdots \simeq a_n = b$ so we can use this same sequence to see that $b \sim a$. Finally if $a \sim b$ and $b \sim c$, then there exists sequences $a_1, a_2, ..., a_n$ and $b_1, b_2, ..., b_m$ such that

$$a = a_1 \simeq a_2 \simeq \cdots \simeq a_n = b$$

and

$$b = b_1 \simeq b_2 \simeq \cdots \simeq b_m = c$$

so

$$a = a_1 \simeq a_2 \simeq \cdots \simeq a_n = b_1 \simeq b_2 \simeq \cdots \simeq b_m = c$$

meaning that the sequence $a_1, a_2, ..., b_1, b_2, ..., b_m$ shows us that $a \sim c$. ■

Definition 5.9. The quotient space $\Pi/\sim$, where Π is equipped with the disjoint union topology, is said to be the space obtained by gluing polyhedra in $\mathcal{P}$ by Φ.

Example. Let us step down a dimension for an example that illustrates gluing. Consider the gluing pattern in E^2 that is shown in Figure 3 where sides of the same color are paired by reflection. Now after the gluing, we can see that the endpoints of the blue segments are related and the same is true for the red segments. Therefore the blue and red segments are S^1 meaning that $\Pi/\sim = S^1 \times S^1$ which is a torus.

Definition 5.10. Let $[x] = \{x_1, x_2, ..., x_n\}$ be a finite equivalence class. Let P_i be the polyhedron in $\mathcal{P}$ that contains x_i. The *solid angle sum* of $[x]$ is

$$\omega[x] = \sum_{i=1}^{n} \omega(P_i, x_i).$$

Definition 5.11. A G-side-pairing Φ for $\mathcal{P}$ is *proper* if and only if each equivalence class of Φ is finite and has a solid angle sum of 4π.

The reason polyhedron gluing is useful is we can create (X, G)-manifolds with this gluing mechanism.

Theorem 5.12. *Let G be a group of isometries of X and let M be a space obtained by gluing together a finite collection $\mathcal{P}$ of disjoint convex polyhedra in X by a proper G-side-pairing Φ. Then M is a 3-manifold with an (X, G)-structure.*

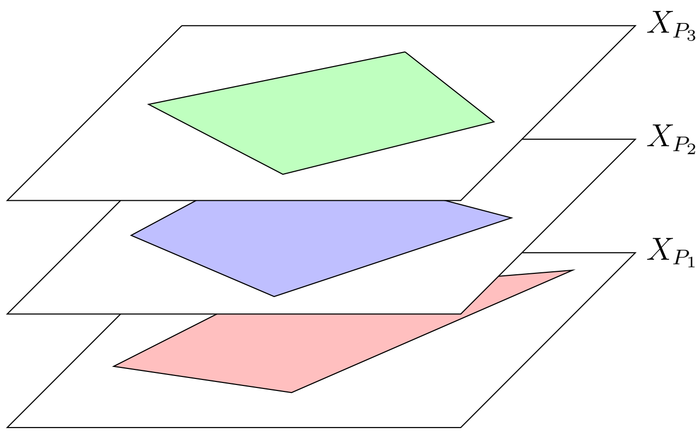


Figure 4. A disjoint set of abstract convex polygons P_1, P_2, and P_3

6. Generalized Gluing

Here we take a digression to discuss necessary concepts needed to prove Theorem 5.12. In this section we work with $X = E^2$, S^2, and H^2 and define a generalized gluing. However, note that this is even though we use the term "generalized," this is completely different from the gluing we discussed before since we are working with 2 dimensional space here.

Definition 6.1. An *abstract convex polygon* P in X is a convex polygon P in X together with a collection $\mathcal{E}$ of subsets of ∂P called the *edges of* P such that

(1) each edge of P is closed, 1 dimensional, and a convex subset of ∂P,

(2) two edges of P meet only along their boundaries,

(3) the union of the edges of P is ∂P, and

(4) the collection $\mathcal{E}$ is a locally finite family of subsets.

Additionally, a *vertex* of an abstract convex polygon is an endpoint of an edge.

For angles, we can use Definition 5.5 but instead of volumes of balls we use areas of disks since we are in 2 dimensional space. We also use a different notation: $\theta(P, x)$

Definition 6.2. A *disjoint set of abstract convex polygons* of X is a set of functions indexed by a set $\mathcal{P}$

$$\Xi = \{\xi_P : X \to X_P \mid P \in \mathcal{P}\}$$

such that

(1) the function ξ_P is a similarity for each $P \in \mathcal{P}$,

(2) the index P is an abstract convex polygon in X_P for each $P \in \mathcal{P}$, and

(3) the polygons in $\mathcal{P}$ are mutually distinct.

Essentially, we are letting each polygon live in its own copy of X so Ξ is like a set of instruction manuals that tells us how to go from the original space to the new copy. Since each polygon lives in its own copy, they must be disjoint. See Figure 4. Now let Ξ be a disjoint set of abstract convex polygons of X and let G be a group of similarities of X.

Definition 6.3. A *G-edge-pairing* for Ξ is the set of functions indexed by the set of all edges of polygons in $\mathcal{P}$

$$\Phi = \{\phi_E : E \in \mathcal{E}\}$$

such that for each edge E of a polygon $P \in \mathcal{P}$

(1) there is a polygon P' in $\mathcal{P}$ such that ϕ_E such $\phi_E : X_{P'} \to X_P$ is a similarity,

(2) the similarity $\xi_P^{-1}\phi_E\xi_{P'}$ is in G,

(3) there is an edge E' of P' such that $\phi_E(E') = E$,

(4) the similarities ϕ_E and $\phi_{E'}$ satisfy the relation $\phi_{E'} = \phi_E^{-1}$, and

(5) the polygons P and $\phi_E(P')$ are defined so that $P \cap \phi_E(P') = E$.

In the same way as Definition 5.7, we can define an equivalence relation on the set $\Pi = \bigcup_{P\in\mathcal{P}} P$ by pairing edge points using elements Φ.

Definition 6.4. Let $[x] = \{x_1, x_2, ..., x_n\}$ be a finite equivalence class. Let P_i be the polygon in $\mathcal{P}$ containing x_i. The *angle sum* of $[x]$ is

$$\theta[x] = \sum_{i=1}^{n} \theta(P_i, x_i).$$

Definition 6.5. A G-edge-pairing Φ for Ξ is *proper* if and only if every equivalence class of Φ is finite and has a angle sum of 2π.

Definition 6.6. The quotient space $\Pi/\sim$, where Π is equipped with the disjoint union topology, is said to be the space obtained by gluing abstract polygons in Ξ by Φ.

Lemma 6.7. *Let G be a group of similarities of X and let M be a space obtained by gluing together a disjoint set Ξ of abstract convex polygons of X by a proper G-edge-pairing Φ. Then M is a 2-manifold with an (X, G)-structure.*

Proof. First we can assume that each polygon in $\mathcal{P}$ has at least one edge. Let $q : \Pi \to M$ be the quotient map and let $\hat{x} \in \Pi$ and $x = \xi_{\hat{P}}^{-1}(\hat{x})$. Now we consider an open neighborhood $U_r(x)$ of $q(x)$ in M and we construct a homeomorphism $f_x : U_r(x) \to B_r(x)$ for values of r that are small enough. Then we shift everything back up to $\hat{x}$.

If $\hat{P} \in \mathcal{P}$ is a polygon containing $\hat{x}$, then x is in polygon $P = \xi_{\hat{P}}^{-1}(\hat{P})$. Next we know that $\hat{x} \in \hat{P}^\circ$, or $\hat{x}$ is in the interior of an edge of $\hat{P}$, or $\hat{x}$ is a vertex of of $\hat{P}$. If $x \in \hat{P}^\circ$, then $[\hat{x}] = \{\hat{x}\}$. If $\hat{x}$ is an interior of an edge, then $[\hat{x}] = \{\hat{x}, \hat{x}'\}$ with $\hat{x} \neq \hat{x}'$ since Φ is proper so the angle sum must be 2π. Finally, if $\hat{x}$ is a vertex, it is on two edges so it gets related to two other points again since Φ is proper.

This means that we can order the elements of $[\hat{x}] = \{\hat{x}_1, \hat{x}_2, ..., \hat{x}_n\}$ such that

$$\hat{x} = \hat{x}_1 \simeq \hat{x}_2 \simeq \cdots \simeq \hat{x}_n \simeq \hat{x}.$$

Now for each i, let $\hat{P}_i \in \mathcal{P}$ contain $\hat{x}_i$ and let $P_i = \xi_{\hat{P}_i}^{-1}(\hat{P}_i)$ which contains $x_i = \xi_{\hat{P}_i}^{-1}(\hat{x}_i)$. Notice that when $n = 1$, we know that $\hat{x} \in \hat{P}^\circ$ so if $n > 1$, we have that $\hat{x}$ is on an edge. This means that there exists a unique edge $\hat{E}_i \in \mathcal{E}$ such that

$$\phi_{\hat{E}_i}(\hat{x}_{i+1}) = \hat{x}_i \text{ for } i = 1, ..., n-1$$

and $\phi_{\hat{E}_n}(\hat{x}_1) = \hat{x}_n$.

Let ϕ_1 be the identity map and $\phi_i = \phi_{\hat{E}_1} \cdots \phi_{\hat{E}_{i-1}}$ for $i = 2, ..., n$. Now let $g_{E_i} = \xi_{\hat{P}_i}^{-1} \phi_{\hat{E}_i} \xi_{\hat{P}_{i+1}}$ for $i = 1, 2, ..., n-1$ and let $g_{E_n} = \xi_{\hat{P}_n}^{-1} \phi_{\hat{E}_n} \xi_{\hat{P}_1}$ which are elements of G where $E_i = \xi_{\hat{P}_i}^{-1}(\hat{E}_i)$. Additionally let $g_i = g_{E_1} \cdots g_{E_{i-1}} \in G$ so

$$g_i = (\xi_{\hat{P}_1}^{-1} \phi_{\hat{E}_1} \xi_{\hat{P}_2})(\xi_{\hat{P}_2}^{-1} \phi_{\hat{E}_2} \xi_{\hat{P}_3}) \cdots (\xi_{\hat{P}_{i-1}}^{-1} \phi_{\hat{E}_{i-1}} \xi_{\hat{P}_i}) = \xi_{\hat{P}_1}^{-1} \phi_i \xi_{\hat{P}_i}$$

for $i = 1, ..., n$. This means that $g_i(x_i) = x$.

Now let r be a positive real such that r is less than a fourth of the distance from x_i to x_j when $i \neq j$ and the distance from x_i to any edge of P_i not containing x_i. Therefore the $P_i \cap B_r(x_i)$ are disjoint for each i.

If $\theta_i = \theta(P_i, x_i)$, then $P_i \cap B_r(x_i)$ is just the sector of angle θ_i of the open disk $B_r(x_i)$. We can backtrack this to x by using g_i so

$$g_i(P_i \cap B_r(x_i)) = g_i(P_i) \cap B_r(x)$$

is a sector of angle θ_i of the open disk $B_r(x)$.

We know that if $n = 1$, then $B_r(x) = g_1(P_1) \cap B_r(x)$ and similarly if $n = 2$, then $B_r(x) = (g_1(P_1) \cap B_r(x)) \cap (g_2(P_2) \cap B_r(x))$. Following the same pattern when $n > 2$, we suspect that

$$B_r(x) = \bigcup_{i=1}^{n} g_i(P_i) \cap B_r(x).$$

The only thing we need to check is if the edges match up.

First we identify these edges which are the edges $g_i(P_i)$ that have a common endpoint of x. We know that E_i' and E_{i+1} are the two edges of P_i that have a common endpoint of x_i which means that $g_{i+1}(E_i')$ and $g_{i+1}(E_{i+1})$ are the two edges of $g_i(P_i)$ that have a common endpoint of x for $i = 1, 2, ..., n-1$.

Now notice that polygons P_i and $g_{E_i}(P_{i+1})$ are on opposite edges of the common edge E_i meaning that polygons $g_i(P_i)$ and $g_{i+1}(P_{i+1})$ are on opposite edges of the common edge $g_i(E_i)$ for $i = 1, 2, ..., n-1$. Additionally,

$$E_i = g_{E_i}(E_i') \implies g_i(E_i) = g_{i+1}(E_i')$$

which means that $g_i(E_i)$ and $g_{i+1}(E_{i+1})$ are the two edges of $g_i(P_i)$ that share a common endpoint of x for $i = 1, 2, ..., n-1$. This means that all the edges match up. See Figure 5.

The polygons P_n and $g_{E_n}(P)$ are on opposite edges of their common edge E_n. Therefore $g_{E_n}^{-1}(P_n)$ and P are on opposite edges of E_n'. Since E_1 and E_n' are the two edges of P who have a common endpoint of x, we have

$$g_n(P_n) = g_{E_n}^{-1}(P_n)$$

so $g_n = g_{E_n}^{-1} \implies g_{E_1} \cdots g_{E_{n-1}} = g_{E_n}^{-1}$. Thus $g_{E_1} \cdots g_{E_n}$ is the identity map.

Let us define

$$U_r(x) = q\left(\bigcup_{i=1}^{n} P_i \cap B_r(x_i)\right)$$

and we claim that it is open in M. First we can see that $P_i \cap B_r(x_i)$ for $i = 1, 2, ..., n$ is open in Π so

$$\bigcup_{i=1}^{n} P_i \cap B_r(x_i) = q^{-1}(U_r(x))$$

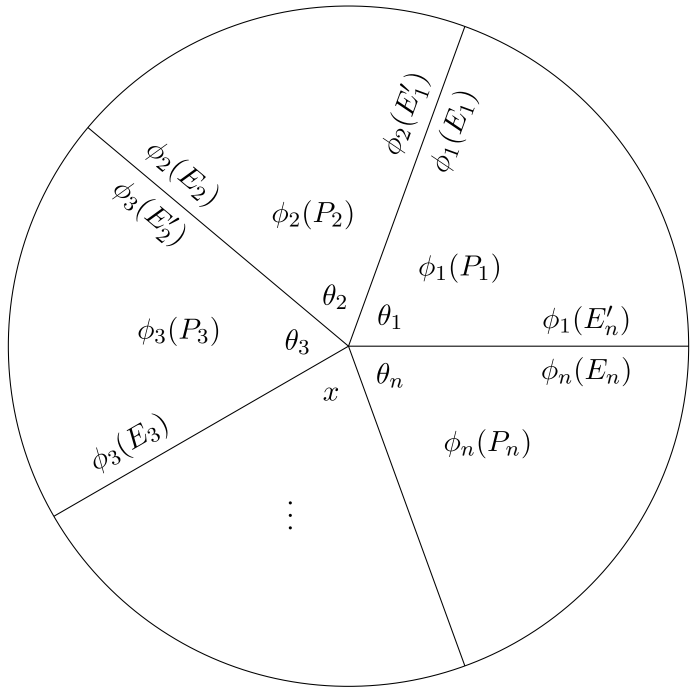


Figure 5. The partition of $B_r(x)$ into sectors of angle θ_i

must also be open in Π. Thus by the definition of a quotient map, are claim must be true. Additionally we can clearly see that $x \in U_r(x)$ so $U_r(x)$ is an open neighborhood of $q(x)$.

Define the function

$$\psi_x : \bigcup_{i=1}^{n} P_i \cap B_r(x_i) \to B_r(x)$$

satisfying $\psi(z) = g_i(z)$ if $z \in P_i \cap B_r(x_i)$. Thus ψ_x induces the continuous bijection

$$f_x : U_r(x) \to B_r(x)$$

with a continuous inverse defined by $f_x^{-1}(z) = q(g_i^{-1}(z))$ whenever $z \in g_i(P_i) \cap B_r(x)$. Therefore f_x is a homeomorphism. Now $B_r(x)$ is an open disk in S^2, H^2, or E^2. Therefore we can perform another homeomorphism to get to an open subset of E^2.

Next for M to be a manifold, it must also be Hausdorff which is what we show now. Recall that a space is Hausdorff if for any two points, there exists disjoint open neighborhoods of them. Let $x, y \in \Pi$ such that $\pi(x) \neq \pi(y)$ in M. Let $[x] = \{x_1, ..., x_n\}$ and $[y] = \{y_1, ..., y_m\}$ be the equivalence classes of x and y, respectively. Then $[x]$ and $[y]$ are disjoint subsets of Π. Let the polygon $P_i \in \mathcal{P}$ contain x_i for $i = 1, 2, ..., n$ and let the polygon $Q_j \in \mathcal{P}$ contain y_j for $j = 1, 2, ..., m$. Let r and s be real numbers such that r is less than a fourth the distance from x_i to x_j when $i \neq j$ and the distance from x_i to an edge of P_i not containing x_i for each i and s is less than a fourth the distance from y_j to y_i when $j \neq i$ and the distance from x_j to an edge of Q_j not containing x_j for each j. This means that

$$U_r(x) = q\left(\bigcup_{i=1}^{n} P_i \cap B_r(x_i)\right)$$

and

$$U_y(x) = q\left(\bigcup_{j=1}^{m} Q_j \cap B_s(y_i)\right).$$

Now we choose r and s small enough such that

$$\bigcup_{i=1}^{n} P_i \cap B_r(x_i) \text{ and } \bigcup_{j=1}^{m} Q_j \cap B_s(y_i)$$

are disjoint in Π. Therefore $U_r(x)$ and $U_y(s)$ are disjoint open neighborhoods of $q(x)$ and $q(y)$, respectively. This completes our proof that M is a 2-manifold.

Now we must prove that M has (X, G)-structure. We show that

$$\{f_x : U_r(x) \to B_r(x)\}$$

is an (X, G)-atlas for M. By definition, we know that $U_r(x)$ is an open connected subset of M and and f_x is a homeomorphism that maps open $U_r(x)$ onto an open subset of X, in this case $B_r(x)$. This means that the $U_r(x)$ are coordinate neighborhoods and the f_x are charts. Furthermore $U_r(x)$ is defined for each point $q(x) \in M$ and a radius of r sufficiently small. Therefore $\{U_r(x)\}$ is an open cover of M. Now, all that remains for us to show is that if $U_r(x)$ and $U_s(y)$ with charts f_x and f_y, respectively, overlap, then the function

$$f_y f_x^{-1} : f_x(U_r(x) \cap U_y(s)) \to f_y(U_r(x) \cap U_s(y))$$

agrees in a neighborhood of each point of its domain with an element of G.

Like before, we have

$$q^{-1}(U_r(x)) = \bigcup_{i=1}^{n} P_i \cap B_r(x_i),$$

and

$$q^{-1}(U_s(y)) = \bigcup_{j=1}^{m} Q_j \cap B_s(y_j).$$

Without loss of generality, assume that $n \leq m$. If $n > 1$, let E_i be the edge of P_i containing x_i and if $m > 1$, let F_j be the edge of Q_j containing y_j. Let $g_1, ..., g_n$ and $h_1, ..., h_n$ be the functions constructed as before for x and y, respectively. Since we have the $1/4$ bound on r and s, there is one index j, call it ℓ, such that the set

$$P \cap B_r(x) \cap Q_j \cap B_s(y_j)$$

is nonempty. Let just prove that $f_y f_x^{-1}$ is the restriction of the element h_ℓ of G.

First assume that $n = 1$. This means that $x \in P^\circ$ and

$$q^{-1}(U_r(x)) = B_r(x).$$

Therefore

$$\begin{aligned}
U_r(x) \cap U_y(s) &= q(B_r(x)) \cap q\left(\bigcup_{j=1}^{m} Q_j \cap B_s(y_j)\right) \\
&= q\left(B_r(x) \cap \bigcup_{j=1}^{m} Q_j \cap B_s(y_j)\right) \\
&= q(B_r(x) \cap B_s(y_\ell)).
\end{aligned}$$

This means that

$$f_x(U_r(x) \cap U_s(y)) = B_r(x) \cap B_s(y_\ell)$$

and

$$f_y(U_r(x) \cap U_s(y)) = h_\ell(B_r(x) \cap B_s(y_\ell)).$$

Hence, the function

$$f_y f_x^{-1} : B_r(x) \cap B_s(y_\ell) \to h_\ell(B_r(x) \cap B_s(y_\ell)$$

is a restriction of h_ℓ.

Now if $n = 2$, then x is in the interior of an edge E of P and x' is in the interior of an edge E' of P' and the set

$$P' \cap B_r(x') \cap Q_j \cap B_s(y_j)$$

is nonempty only when $j \equiv \ell - 1$ or $\ell + 1 \pmod m$. Now without loss of generality, we may assume the latter. This means that $P = Q_\ell$, $P' = Q_{\ell+1}$, and $E = F_\ell$. Therefore,

$$\begin{aligned} U_r(x) \cap U_s(y) &= q[(P \cap B_r(x)) \cup (P' \cap B_r(x'))] \cap q\left[\bigcup_{j=1}^{m} Q_j \cap B_s(y_j)\right] \\ &= q\left[\bigcup_{j=1}^{m}(P \cap B_r(x) \cap Q_j \cap B_s(y_j)) \cup \bigcup_{j=1}^{m}(P' \cap B_r(x') \cap Q_j \cap B_s(y_j))\right] \\ &= q[(P \cap B_r(x) \cap B_s(y_\ell)) \cup (P' \cap B_r(x') \cap B_s(y_{\ell+1}))] \end{aligned}$$

Thus,

$$\begin{aligned} f_x(U_r(x) \cap U_s(y)) &= g_E(P \cap B_r(x) \cap B_s(y_\ell)) \cup g_E(P' \cap B_r(x') \cap B_s(y_{\ell+1})) \\ &= (P \cap B_r(x) \cap B_s(y_\ell)) \cup (g_E(P') \cap B_r(x) \cap B_s(y_\ell)) \\ &= B_r(x) \cap B_s(y_\ell). \end{aligned}$$

and

$$\begin{aligned} f_y(U_r(x) \cap U_s(y)) &= h_\ell(P \cap B_r(x) \cap B_s(y_l)) \cup h_{\ell+1}(P' \cap B_r(x') \cap B_s(y_{\ell+1})) \\ &= h_\ell[(P \cap B_r(x) \cap B_s(y_\ell)) \cup g_E(P' \cap B_r(x') \cap B_s(y_{\ell+1}))] \\ &= h_\ell[(P \cap B_r(x) \cap B_s(y_\ell)) \cup (g_E(P') \cap B_r(x) \cap B_s(y_\ell))] \\ &= h_\ell(B_r(x) \cap B_s(y_\ell)). \end{aligned}$$

Now on the set

$$P \cap B_r(x) \cap B_s(y_\ell),$$

we can see that the map $f_y f_x^{-1}$ is a restriction of h_ℓ. Similarly, on the set

$$g_E(P' \cap B_r(x') \cap B_s(y_{\ell+1})),$$

the map $f_y f_x^{-1}$ is the restriction of $h_{\ell+1} g_E^{-1} = h_\ell$. Thus the map

$$f_y f_x^{-1} : B_r(x) \cap B_s(y_\ell) \to h_\ell(B_r(x) \cap B_s(y_\ell))$$

is the restriction of h_ℓ.

Finally, assume that $n > 2$. This means that both x and y are vertices. Since $U_r(x)$ and $U_s(y)$ overlap, we have $q(x) = q(y)$ resulting from the bounds for r and s. Therefore $x = y_\ell$. Let $t = \min(r, s)$. Then

$$U_r(x) \cap U_s(y) = U_t(x),$$
$$f_x(U_t(x)) = B_t(x),$$

and

$$f_y(U_t(x)) = B_t(y).$$

Now $x_i \equiv y_{\ell+i-1}$ or $y_{\ell-i-1} \pmod{n}$. Without loss of generality, we may assume the former. Then

$$P_i \equiv Q_{\ell+i-1} \text{ and } E_i \equiv F_{\ell+i-1} \pmod{n}.$$

Notice that

$$\begin{aligned} g_i &= g_{E_1} \cdots g_{E_{i-1}} \\ &\equiv g_{F_\ell} \cdots g_{F_{\ell+i-2}} \\ &\equiv h_\ell^{-1} h_{\ell+i-1} \pmod{n} \end{aligned}$$

which means that

$$h_{\ell+i-1} \equiv h_\ell g_i \pmod{n}.$$

Now since

$$B_t(x) = \bigcup_{i=1}^{n} g_i(P_i) \cap B_t(x),$$

the map $f_y f_x^{-1}$ is the restriction of

$$h_{\ell+i-1} g_i^{-1} = (h_\ell g_i) g_i^{-1} = h_\ell$$

on the set $g_i(P_i) \cap B_t(x)$ for each $i = 1, ..., n$. Therefore the map

$$B_t(x) \to B_t(y)$$

is the restriction of h_ℓ. Therefore we have completed the proof that $\{f_x\}$ is an (X, G)-atlas for M. ■

7. Proof of Theorem 5.12

For convenience, we restate the theorem here.

Theorem 7.1. *Let G be a group of isometries of X and let M be a space obtained by gluing together a finite collection $\mathcal{P}$ of disjoint convex polyhedra in X by a proper G-side-pairing Φ. Then M is a 3-manifold with an (X, G)-structure.*

Proof. Without loss of generality, we can assume that each polyhedron in $\mathcal{P}$ has at least one side. Let x be a point Π and let $[x] = \{x_1, ..., x_n\}$. Let $P_i \in \mathcal{P}$ be the polyhedron containing x_i for each i. If x_i is contained in a side of P_i, then $m \geq 2$. Now we can let $\delta(x)$ be the minimum distance from x_i to x_j for $i \neq j$ and distance from x_i to any side of P_i not containing x_i for each i.

Let r be a real such that $0 < r < \delta(x)/2$. Then for each i, the set $P_i \cap S_r(x_i)$ is a polygon in the sphere $S_r(x_i)$ and these polygons are disjoint. Now the side-pairing Φ restricts to a proper $\mathrm{I}(S^2)$-side-pairing of $\{P_i \cap S_r(x_i)\}$. Let $\Sigma_r(x)$ be the space obtained by gluing together the polygons. By Lemma 6.7, the space $\Sigma_r(x)$ has spherical structure. Since $\Sigma_r(x)$ is compact, connected, and $\omega[x] = 4\pi$, we can see that $\Sigma_r(x)$ is a 2-sphere

Let $q : \Pi \to M$ be the quotient map. For each i, the restriction of q to the polygon $P_i \cap S_r(x_i)$ extends to an isometry

$$\chi_i : S_r(x_i) \to \Sigma_r(x).$$

Additionally, for each i and j, the isometry

$$\chi_j^{-1}\chi_i : S_r(x_i) \to S_r(x_j)$$

extends to a unique isometry g_{ij} of X where $g_{ij}(x_i) = x_j$.

Suppose that $g_S \in \Phi$ pairs the side $S' \cap S_r(x_i)$ of $P_i \cap S_r(x_i)$ to the side $S \cap S_r(x_j)$ of $P_j \cap S_r(x_j)$. This means that $\chi_j^{-1}\chi_i$ agrees with g_S on the set $S' \cap S_r(x_i)$. Therefore $\chi_j^{-1}\chi_i$ agrees with g_S on the great circle $\langle S'\rangle \cap S_r(x_i)$. Thus g_{ij} agrees with g_S on $\langle S'\rangle$. Since g_{ij} and g_S both map $P_i \cap S_r(x_i)$ to the opposite side of $\langle S\rangle$ from $P_j \cap S_r(x_j)$, we see that $g_{ij} = g_S$.

Now if

$$x_i = x_{i_1} \simeq x_{i_2} \cdots \simeq x_{i_p} = x_j,$$

then

$$\chi_j^{-1}\chi_i = (\chi_{i_p}^{-1}\chi_{i_{p-1}})(\chi_{i_{p-1}}^{-1}\chi_{i_{p-2}})\cdots(\xi_{i_2}^{-1}\chi_{i_1})$$

which means that

$$g_{ij} = g_{i_{p-1}i_p} g_{i_{p-2}i_{p-1}} \cdots g_{i_1 i_2}$$

and $g_{i_1i_2}, ..., g_{i_{p-1}i_p} \in \Phi$ so $g_{ij} \in G$ for each i, j.

Define

$$U_r(x) = \bigcup_{i=1}^{n} q(P_i \cap B_r(x_i)).$$

Since

$$q^{-1}(U_r(x)) = \bigcup_{i=1}^{n} P_i \cap B_r(x_i)$$

is open in Π, we have that $U_r(x)$ is open in M.

Suppose $x = x_k$ and let

$$\psi : \bigcup_{i=1}^{n} P_i \cap B_r(x_i) \to B_r(x)$$

be a function defined by $\psi_x(z) = g_{ik}(z)$ if $z \in P_i \cap B_r(x_i)$. Now if $g_S(x_i) = x_j$, then $g_S = g_{ij}$. Let y be a point in $S \cap B_r(x_j)$ and let $y' = g_S^{-1}(y)$ so y' is a point of $S' \cap B_r(x_i)$. Since

$$\chi_k^{-1}\chi_i = (\chi_k^{-1}\chi_j)(\chi_j^{-1}\chi_i),$$

we have that $g_{ik} = g_{jk}g_{ij}$. Thus

$$\psi_x(y) = g_{jk}(y) = g_{jk}g_S(y') = g_{ik}(y') = \psi_x(y').$$

This results in ψ_x inducing a continuous function

$$\phi_x : U_r(x) \to B_r(x).$$

For each t such that $0 < t < r$, the map ϕ_x restricts to the isometry

$$\chi_k^{-1} : \Sigma_t(x) \to S_t(x).$$

Thereeforе ϕ_x is a bijection with the continuous inverse

$$\phi_x^{-1}(z) = qg_{ik}^{-1}(z)$$

if $z \in g_{ik}(P_i \cap B_r(x_i))$ meaning that ϕ_x is a homeomorphism. We can consider the same argument as Lemma 6.7 to show that M is Hausdorff so this completes our proof that M is a 3-manifold.

Now all that remains to show is that M has (X, G)-structure. Specifically, we show that

$$\{\phi_x : U_r(x) \to B_r(x) : x \in \Pi \text{ and } r < \delta(x)/3\}$$

is an (X, G)-atlas for M. By definition, we know that $U_r(x)$ is an open connected subset of M and ϕ_x is a homeomorphism. Furthermore $U_r(x)$ is defined for each point $q(x)$ of M and a radius r that is sufficiently small. This means that $\{U_r(x)\}$ is an open cover of M.

Suppose $U_r(x)$ and $U_s(y)$ overlap and $r < \delta(x)/3$ and $s = \delta(y)/3$. Let $F(x)$ be the face of the polyhedron in $\mathcal{P}$ that contains x in its interior. Without loss of generality, we may assume that

$$\dim F(x) \geq \dim F(y).$$

Like before,

$$q^{-1}(U_r(x)) = \bigcup_{i=1}^{n} P_i \cap B_r(x_i)$$

and

$$q^{-1}(U_s(y)) = \bigcup_{j=1}^{m} Q_j \cap B_s(y_j).$$

Now for some i and j, the set $P_i \cap B_r(x_i)$ mmets $Q_j \cap B_s(y_j)$. After reindexing, we can assume that $P_1 \cap B_r(x_1)$ meets $Q_1 \cap B_s(y_1)$ so $P_1 = Q_1$ and by the triangle inequality, $d(x_1, y_1) < r + s$.

Suppose y_1 is not in a side of P_1 containing x_1. This means that $s < d(x_1, y_1)/3$ so $r < d(x_1, y_1)/3$ which is a contradiction. Therefore x_1 is in every side of P_1 that contains y_1. Thus $F(x_1)$ is a proper face of $F(y_1)$ which is another contradiction. This means that y_1 is in every side of P_1 that contains x_1. Consequently, for each i, the set $P_i \cap B_r(x_i)$ meets $Q_j \cap B_s(y_j)$ for some j.

Now we claim that the set $P_i \cap B_r(x_i)$ meets $Q_j \cap B_s(y_j)$ for one index j. For the sake of contradiction, suppose that $P_i \cap B_r(x_i)$ meets $Q_j \cap B_s(y_j)$ and $Q_k \cap B_s(y_k)$ with $j \neq k$. Then $P_i = Q_j = Q_k$. Since y_j and y_k are in every side of P_i that contains x_i, we see that $F(y_j)$ and $F(y_k)$ are faces of $F(x_i)$. Furthermore, $F(y_i)$ and $F(y_k)$ must be distinct. Therefore $F(y_i)$ and $F(y_k)$ are proper faces of $F(x_i)$. This means that

$$r < d(x_i, y_i)/3, \; r < d(x_i, y_k)/3, \text{ and } s < d(y_j, y_k)/3.$$

Now by the triangle inequality,

$$\begin{aligned} d(x_i, y_j) + d(x_i, y_k) &< (r + s) + (r + s) \\ &< \frac{d(x_i, y_j)}{3} + \frac{d(x_i, y_k)}{3} + \frac{2 \cdot d(y_j, y_k)}{3} \\ &< d(x_i, y_j) + d(x_i, y_k) \end{aligned}$$

which is a contradiction.

Next we claim that the set $Q_j \cap B_s(y_j)$ meets $P_i \cap B_r(x_i)$ for one index i. Again for the sake of contradiction, suppose that $Q_j \cap B_s(y_j)$ meets $P_i \cap B_r(x_i)$ and $P_k \cap B_r(x_k)$ with $i \neq k$. Then $P_i = Q_j = P_k$. Since y_j is in every side of P_i that contains x_i or x_k, we know that $F(y_j)$ is a face of $F(x_i)$ and $F(x_k)$. Furthermore, $F(x_i)$ and $F(x_k)$ must be distinct. Therefore, we have

$$r < \frac{d(x_i, y_j)}{3} < \frac{r + s}{3}.$$

This means that $r < s/2$. Since $s < \delta(y)/3$, we see that $r < \delta(y)/6)$. Notice that

$$d(x_i, y_j) < r + s < \delta(y)/2 \text{ and } d(x_k, y_j) < r + s < \delta(y)/2.$$

By the definition of $U_{r+s}(y)$, we see that q maps $P_i \cap B_{r+s}(y_j)$ injectively to M. Since x_i and x_k are in $P_i \cap B_{r+s}(y_j)$, we have a contradiction. This means that we can reindex $[y]$ such that $P_i \cap B_r(x_i)$ meets $Q_j \cap B_s(y_i)$ for $i = 1, ..., m$. Then $P_i = Q_i$ for each i.

Now let g_{ij} and h_{ij} be the elemetns of G constructed as before for x and y. If g_S pairs the side $S' \cap S_r(x_i)$ of $P \cap S_r(x_i)$ to the side of $S \cap S_r(x_j)$ of $P_j \cap S_r(x_j)$, then $g_S = g_{ij}$ and $g_S(x_i) = x_j$. Thus $x_i \in S'$. Now since $P_i \cap B_r(x_i)$ meets $P_i \cap B_s(y_i)$, we have that y_i is also in S'. Notice that $g_S(P_i \cap B_r(x_i))$ meets $g_s(P_i \cap B_s(y)i)$. This means that $P_j \cap B_r(x_j)$ meets $P_j \cap B_s(g_s(y_i))$. Therefore $g_s(y_i) = y_j$ which means that $g_{ij} = h_{ij}$.

Suppose that

$$x_i = x_{i_1} \simeq x_{i_2} \simeq \cdots \simeq x_{i_p} = x_j$$

like before. This means that

$$y_i = y_{i_1} \simeq y_{i_2} \simeq \cdots \simeq y_{i_p} = y_j$$

and

$$g_{ij} = g_{i_{p-1}i_p} g_{i_{p-2}i_{p-1}} \cdots g_{i_1 i_2} = h_{i_{p-1}i_p} h_{i_{p-2}i_{p-1}} \cdots h_{i_1 i_2} = h_{ij}.$$

Next, notice that

$$\begin{aligned} U_r(x) \cap U_s(y) &= q\left(\bigcup_{i=1}^{n} P_i \cap B_r(x_i)\right) \cap q\left(\bigcup_{j=1}^{m} Q_j \cap B_s(y_j)\right) \\ &= q\left(\left(\bigcup_{i=1}^{n} P_i \cap B_r(x_i)\right) \cap \left(\bigcup_{j=1}^{m} Q_j \cap B_s(y_j)\right)\right) \\ &= q\left(\bigcup_{i=1}^{n}\bigcup_{j=1}^{m} (P_i \cap B_r(x_i) \cap Q_j \cap B_s(y_j))\right) \\ &= q\left(\bigcup_{i=1}^{n} P_i \cap B_r(x_i) \cap B_s(y_i))\right). \end{aligned}$$

If $x = x_k$ and $y = y_\ell$, then

$$\phi_x(U_r(x) \cap U_s(y)) = \bigcup_{i=1}^{n} g_{ik}(P_i \cap B_r(x_i) \cap B_s(y_i))$$

and

$$\phi_y(U_r(x) \cap U_s(y)) = \bigcup_{i=1}^{n} h_{i\ell}(P_i \cap B_r(x_i) \cap B_s(y_i)).$$

Now on the set

$$g_{ik}(P_i \cap B_r(x_i) \cap B_s(y_i)),$$

the map $\phi_y \phi_x^{-1}$ is the restriction of

$$h_{i\ell} g_{ik}^{-1} = h_{i\ell} h_{ik}^{-1} = h_{i\ell} h_{ki} = h_{k\ell}$$

for each $i = 1, ..., n$. Therefore $\phi_y \phi_x^{-1}$ is the restriction to $h_{k\ell}$ and $\phi_y \phi_x^{-1}$ agrees with an element of G. This completes our proof that $\{\phi_x\}$ is an (X, G)-atlas for M. ∎

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