ORIGAMI CONSTRUCTABLE NUMBERS

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Abstract. This paper presents an exposition of origami's expansion to the constructible field. We first define and employ basic field theory and axioms based on sets of manipulations to define the constructible field. We explore certain constructions through the set of single-fold origami constructions, such as the Beloch Fold, $\sqrt[3]{2}$, doubling the cube, and trisecting the angle. Over the course of these sections, I present proofs upon the constructibility of objects. Lastly, this paper presents a method to solve polynomials with origami.

1. Introducing The Fold

It's 478BC ancient Greece and a plague sent by Apollo has devastated the small island of Delos. The oracle has presented a problem that, once solved, would end the plague. Apollo must have his cubic altar's volume doubled.

In a world of solely lines and circles, you may only use a straightedge and a compass. Soon after, another problem arises: given a circle of area A, construct a square of the same area, A. As countless people tired over these problems over the course of 2000 years, there comes a breakthrough.

It isn't possible.

But, being a true mathematician, you must question why. You begin by learning the basics of field theory, after noticing parallels between constructible numbers and fields.

2. Field Theory

Definition 2.1 (Field)**.** A field, F, is a set of elements that satisfy a number of field axioms:

(1) The set is closed under multiplication and addition.
(2) Both addition and multiplication must be commutative and associative.
(3) The distributive property holds for addition and multiplication.
(4) There is an additive identity for every element as well as a multiplicative identity for every element.
(5) There is an additive inverse for every element as well as a multiplicative inverse for every element except 0.

Some examples of fields are the rational ($\mathbb{Q}$), real ($\mathbb{R}$), and complex ($\mathbb{C}$) numbers.

However, some fields contain all the numbers from another field:

Definition 2.2 (Subfield)**.** A subfield E of F is a field consisting of a subset of the elements in field F satisfying the field axioms of F. In other words, subfield E of F is a subset of F containing the multiplicative identity of F and is closed under multiplication, addition, the multiplicative inverse and additive inverse across all nonzero elements of F.
For example, the rational numbers are a subfield of the real numbers, which in turn, are a subfield of the complex numbers.

In order to get from a subfield to the original field, we need to introduce a field extension:

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Definition 2.3 (Field Extensions)**.** A field extension is a the smallest extension of a field when an element is added such that the new field satisfies all field axioms.
A finite field extension is the field generated by adjoining $x_1, \ldots, x_n$ to the original field.
For example, a field extension of the real field, $\mathbb{R}$, by the imaginary number i results in the complex field, $\mathbb{C}$.

A generalized form of a field, of sorts, is a group:

Definition 2.4 (Groups)**.** A group is a set of finite or infinite elements that satisfy the properties of closure, the additive property, the inverse property, and the identity property. A group is a generalized form of a field: the closure property holds under any binary operation, not necessarily addition and multiplication only. An example of a group is the integers with the operation of addition.

Similar to fields, we have subgroups:

Definition 2.5 (Subgroups)**.** A subgroup is a subset of a group closed under the group operation and the inverse operation. An example of a subgroup of rationals under addition is integers under addition.

Definition 2.6 (Abelian)**.** An Abelian group is a group in which applying the group operation is independent of the order of the elements. For example, in the real numbers, + is abelian.

Similarly, we have 2 types of fields:

Definition 2.7 (Algebraic, Transcendental)**.** For 2 fields K and L, where L is a field extension of K, element α of L is called algebraic over K if α is a root of polynomial $p(x)$ in $K[x]$. Element α of L is transcendental over K if α is not a root of polynomial $p(x)$ in $K[x]$

From here, we can begin to explore what numbers we can construct with a straightedge and compass in terms of fields.

3. Straightedge and Compass Constructions

Straightedge and Compass constructions can be synthesized into some manipulations that can be achieved. We can repeatedly apply these manipulations to construct different points and lines.

(1) Creating the line through two existing points
(2) Creating the circle through one point with center another point
(3) Creating the point which is the intersection of two existing, non-parallel lines
(4) Creating the one or two points in the intersection of a line and a circle (if they intersect)
(5) Creating the one or two points in the intersection of two circles (if they intersect).

We will now be attempting to characterize all constructible points using field theory:

Proposition 3.1. *For some field F a subfield of $\mathbb{R}$ and some point (a,b) one step away (one straightedge and compass axiom away) from $F \times F = \{(a, b) \mid a, b \in F\}$,*

$$[F(a, b) : F] = 2^m$$

for some $m \in \mathbb{N} \cup 0$.

Proof. Constructible point (a,b) is the intersection of two distinct figures. We will be taking the last 3 manipulations (3-5) from the list above into consideration.

We now take 3 cases:
Case 1: Intersection of 2 distinct lines defined by 2 pairs of points: $A(x_a, y_a)$, $B(x_b, y_b)$ and $C(x_c, y_c), D(x_d, y_d)$, with A, B, C, and D contained in $F \times F$. Let us label the intersection point of the 2 lines as $Q(a, b)$. We have the equations:

$$(x_b - x_a)(b - y_a) = (y_b - y_a)(a - x_a)$$

$$(x_d - x_c)(b - y_c) = (y_d - y_c)(a - x_c).$$

Once solved, these equations will provide a solution to a and b in terms of field F. Therefore, $F(a, b) = F$:

$$[F(a, b) : F] = [F : F] = 2^0$$

Case 2: Intersection of a line defined by 2 points: $A(x_a, y_a)$, $B(x_b, y_b)$ and a circle with radius CD and center E: $C(x_c, y_c), D(x_d, y_d)$, and $E(x_e, y_e)$ with A, B, C, D, and E contained in $F \times F$. Let us label the intersection point of the 2 lines as $Q(a, b)$. We can solve for the radius r:

$$r = \sqrt{(x_c - x_d)^2 + (y_c - y_d)^2}$$

$$(x_b - x_a)(b - y_a) = (y_b - y_a)(a - x_a)$$

$$(a - x_e)^2 + (b - y_e)^2 = r^2$$

When we solve for a or b, we get an expression in terms of other elements in F. We have polynomials with roots including a and b in $F[x]$ of a degree less than or equal to 2. In each polynomial ring, the polynomials are either irreducible, or have a factor of which a and b are roots of. Therefore, we have:

$$[F(a) : F] \leq 2, [F(a, b) : F(a)] \leq 2$$

This gives us: $[F(a) : F], [F(a, b) : F(a)] \in 1, 2$, implying that $[F(a) : F]$ and $[F(a, b) : F(a)] \leq 2$ are both powers of 2:

$$[F(a, b) : F] = [F(a, b) : F(a)][F(a) : F] = 2^m$$

for some $m \in \mathbb{N} \cup 0$.

Case 3: Intersection of a circle with radius AB: $A(x_a, y_a)$, $B(x_b, y_b)$ centered on $C(x_c, y_c)$ and a circle with radius DE and center G: $D(x_d, y_d), E(x_e, y_e)$, and $G(x_g, y_g)$ with A, B, C, D, E, and G contained in $F \times F$. Let us label the intersection point of the 2 lines as $Q(a, b)$. Additionally, the radius of the circle centered at C is r_c and the radius of the circle centered at G is r_g. Again, we have: $r_c^2, r_g^2 \in F$,

$$(a - x_c)^2 + (b - y_c)^2 = r_c^2$$

$$(a - x_g)^2 + (b - y_g)^2 = r_g^2$$

We can express the first equation in the form

$$a^2 + b^2 + l_c a + m_c b + n_c = 0$$

for $l_c, m_c, n_c \in F$. Similarly, we can express the second equation in terms of l_g, m_g, and n_g. When we subtract one equation from the other, we have that (a, b) lies on the line:

$$(l_c - l_g)x + (m_c - m_g)y + (n_c - n_g) = 0.$$

We know that the circles are distinct, so at least one of l_c and l_g, m_c, and m_g, or n_c and n_g cannot equal 0. Therefore, we have the above equation is a line. Additionally, all the coefficients are in F, so (a, b) must lie on a line in F. Therefore, we have:

$$[F(a, b) : F] = [F(a, b) : F(a)][F(a) : F] = 2^m$$

for some $m \in \mathbb{N} \cup 0$.

For all 3 cases, we have that

$$[F(a, b) : F] = 2^m$$

for some $m \in \mathbb{N} \cup 0$. □

We now expand upon the previous proposition (3.1) to get a generalization true for all straightedge and compass constructible points.

Proposition 3.2. *All constructible points lie on a 2^n degree field extension of $\mathbb{Q}$.*

Proof. Let F be a subfield of $\mathbb{R}$ and let point $(a, b) \in \mathbb{R}^2$ constructible from $F \times F$. We know there exists points $P_1, P_2, P_3, \dots, P_n \in \mathbb{R}^2$ such that each point $P_i 1 \leq i \leq n$ is constructible in one step from $(F \times F) \cup \{P_1, P_2, \dots, P_n\}$. Let $P_1 = (a_i, b_i)$ and $K_i = K_{i-1}(a_i, b_i)$. We have that $K_0 = F$. Therefore, we have, for some $0 \leq j \leq n-1$:

$$(F \times F \cup \{P_1, P_2, \dots, P_n\} \subseteq K_j \times K_j$$

It directly follows that

$$(F \times F) \cup \{P_1, P_2, \dots, P_n\} \subseteq (K_i \times K_i) \cup \{P_{j+1}\}.$$

We simplify the right side to get

$$(F \times F) \cup \{P_1, P_2, \dots, P_n\} \subseteq (K_j(a_{j+1}, b_{j+1}) \times K_j(a_{j+1}, b_{j+1}) = K_{j+1} \times K_{j+1}.$$

Using induction for $0 \leq j \leq n$, we have:

$$(F \times F) \cup \{P_1, P_2, \dots, P_n\} \subseteq K_{j+1} \times K_{j+1}.$$

We know that P is constrictible from one step from $(F \times F) \cup \{P_1, P_2, \dots, P_n\}$ and is, therefore, constructible in one step from $K_{j+1} \times K_{j+1}$. From Proposition 3.1, we know

$$[K_i : K_{i-1}] = [K_{i-1}(a_i, b_i) : K_{i_1}] = 2^m$$

for some natural number m. From this, we can expand the left hand side:

$$[K_n : F] = [K_n : 0]$$
$$[K_n : F] = [K_n : K_{n-1}][K_{n-1} : K_{n-2}] \dots [K_1 : K_0]$$
$$[K_n : F] = 2^{m_n} 2^{m_{n-1}} \dots 2_1^m$$
$$[K_n : F] = 2^m$$

We have that $F = K_0 \subseteq K_1 \subseteq K_2 \subseteq \cdots \subseteq K_n$, so $f \subseteq K_n$. Again, we have:

$$2^m = [K_n : F] = [K_n : F(a, b)][F(a, b) : F].$$

Now, we have $[F(a, b) : F]$ divides 2^m. Therefore, $[F(a, b) : F]$ must be 2^k for some k, $0 \leq k \leq m$. □

Now we know which points are straightedge and compass constructible and which are not.

There are 3 main constructions that are impossible to achieve (Sections 6-8) with simply a straightedge and compass. These constructions are the following:

(1) Given a cube of volume v, construct a cube with a volume $2v$.
(2) Given angle PQR measuring $3n°$, trisect the angle into 3 angles measuring $n°$.
(3) Given a square of area a, construct a circle with the same area as the square.

We will attempt to solve these problems with origami.

4. Origami Constructions

Employing origami, we are able to form more constructions:

(1) Given two distinct points p_1 and p_2, there is a unique fold that passes through both of them.
(2) Given two distinct points p_1 and p_2, there is a unique fold that places p_1 onto p_2.
(3) Given two lines l_1 and l_2, there is a fold that places l_1 onto l_2.
(4) Given a point p_1 and a line l_1, there is a unique fold perpendicular to l_1 that passes through point p_1.
(5) Given two points p_1 and p_2 and a line l_1, there is a fold that places p_1 onto l_1 and passes through p_2.
(6) Given two points p_1 and p_2 and two lines l_1 and l_2, there is a fold that places p_1 onto l_1 and p_2 onto l_2.
(7) Given one point p and two lines l_1 and l_2, there is a fold that places p onto l_1 and is perpendicular to l_2.

Proposition 4.1. *Let $r \in \mathbb{R}$. Then r is in the origami constructible group if and only if there exists fields $\mathbb{Q} = F_0 \subset F_1 \subset F_2 \subset \cdots \subset F_n \subset \mathbb{R}$ such that $r \in F_n$ and $[F_i : F_{i-1}] = 2$ or 3 for all $1 \leq i \leq n$.*

Proof. Suppose $\mathbb{Q} = F_0 \subset F_1 \subset F_2 \subset \cdots \subset F_n \subset \mathbb{R}$ and $[F_i : F_{i-1}] = 2$ or 3. It follows that $F_i = F_{i-1}[\sqrt{d_i}]$ or $F_i = F_{i-1}[\sqrt[3]{d_i}]$. Note that all rational numbers are constructible with a compass and straightedge. Assume that all elements of F_n are origami constructible for $0 \leq n \leq k-1$.

Then for some $d_k \in F_{k-1}$, it is well known that $\sqrt{d_k}$ and $\sqrt[3]{d_k}$ are also constructible. Therefore, the elements in $F_k = F_{k-1}[\sqrt{d_k}]$ and $F_k = F_{k-1}[\sqrt[3]{d_k}]$ are origami constructible. We use induction to prove that any $r \in F_n$ is origami constructible. □

Proposition 4.2. *If r is origami constructible then $[\mathbb{Q}(r); \mathbb{Q}] = 2^a 3^b$ for integers $a, b \geq 0$.*

Proof. We can expand upon proposition 4.1 using the tower rule[5]:

$$[F_n : \mathbb{Q}] = [F_n : F_{n-1}][F_{n-1} : F_{n-2}] \dots [F_1 : F_0] = 2^c 3^d$$

in which $c + d = n$.

We also have $\mathbb{Q} \subset \mathbb{Q}(r) \subset F_n$, so we apply the tower rule:

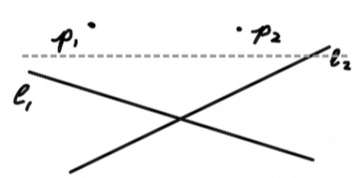
$$[\mathbb{Q}(r) : \mathbb{Q}][F_n : \mathbb{Q}] = 2^c 3^d.$$

If r is an origami constructible number, then $[\mathbb{Q}(r) : \mathbb{Q}] = 2^a 3^b$ for integers $a, b \geq 0$ □

5. The Beloch Fold and Square

The fold that sets origami constructions apart from straightedge and compass constructions is the Beloch Fold:

Beloch Fold: Given two points A and B and two lines l_1 and l_2, there exists a fold F placing A onto l_1 and B onto l_2 simultaneously.



We first need to understand what we are doing when we fold point A onto line l_1. The crease produced is the tangent of the parabola formed with focus A and directrix l_1.

We can visualize this by taking a piece of paper and labeling an arbitrary point A and line l_1. We start making folds placing point A onto arbitrary points on l_1. As we make more folds, the outline of a parabola begins to form.

To prove the tangency of the crease, we can take any point X on line l_1 and label the intersection point of the normal to l_1 and crease X'. We then compare the distances X'A and X'X. They turn out to be equal. By the definition of a parabola, point X' must be on the parabola. Any other point on the crease will only be equidistant between X and A and will therefore not be on the parabola. The crease is tangent to the parabola at point X'.

Proposition 5.1. *The equation of a crease line f tangent to the parabola formed with focus $P(0,1)$ and line l with equation $y = -1$ is $y = \frac{t}{2}x - \frac{t^2}{4}$.*

Recall from section 5 that a point-line pair crease is tangent to a parabola with a focus at the point and a directrix at the line.

We can define point $P'(t, -1)$ where $t \in \mathbb{R}$ as the point where point P was folded onto line l. We know that crease f is the perpendicular bisector of PP'. We have the slope of line PP' as $-\frac{2}{t}$ and the midpoint of line PP' as $(\frac{t}{2}, 0)$.

We can solve for the equation of the crease line, f as we know the slope and a point on the line:

$$f : y = \frac{t}{2}x - \frac{t^2}{4}$$

Definition 5.1 (Beloch Square)**.** Given two points A and B and two lines l_1 and l_2, a Beloch square is a square XYWZ such that X and Y lie on l_1 and l_2 respectively, A lies on line XZ and B lies on line YW.

To create a square, we must have 2 sets of parallel sides set at 90° to adjacent sides. Let x denote the shortest distance between A and line l_1 and let l_1' be a line parallel to line l_1 set a distance of x away from line l_1 such that l_1 lies between A and l_1'. (Figure 1)

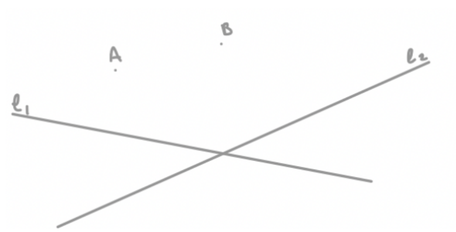


Figure 1. Construction of Beloch Square

Likewise, let y denote the shortest distance between B and line l_2 and let l_2' be a line parallel to line l_2 set a distance of y away from line l_2 such that l_2 lies between B and l_2'. (Figure 2)

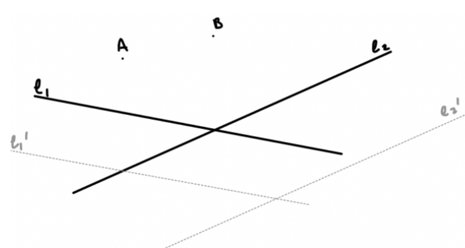


Figure 2. Construction of Beloch Square

We then employ the Beloch fold, folding A onto l_1' and B onto l_2' to create A' and B', respectively. This crease is the perpendicular bisector of AA' and BB'. If we find the midpoints of AA' and BB', X and Y respectively, we know that X lies on l_1 and Y on l_2 using the definitions of l_1 and l_2. (Figure 3) 4

We then construct vertices W and Z by moving a distance of XY past X and Y, respectively. (Figure 4)

The Beloch square also provides us with the construction of the cube root of any number. Let us define l_1 as the y-axis, l_2 as the x-axis, A=$(-1, 0)$, and B=$(0, -k)$. We construct lines l_1' and l_2' as $x = 1$ and $y = k$ respectively. We use the Beloch fold to fold A and B onto l_1' and l_2', respectively,

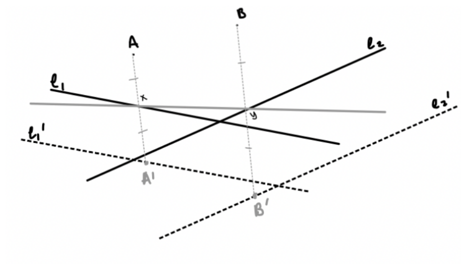


FIGURE 3. Construction of Beloch Square

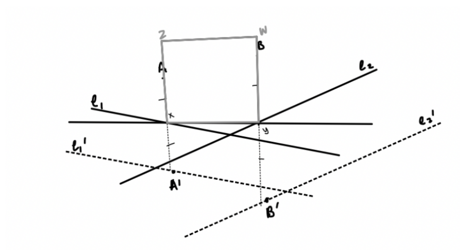


FIGURE 4. Construction of Beloch Square

to find X and Y. X is the intersection of the crease and l_1 and Y is the intersection of the crease and l_2. Let O be the origin. We then get that:

$$\triangle OAX \sim \triangle OXY \sim \triangle OBY.$$

From that, we have:

$$\frac{OX}{OA} = \frac{OY}{OX} = \frac{OB}{OY}.$$

We know that OA = 1 and OB = k. Plugging in, we get $OX = \frac{OY}{OX} = \frac{k}{OY}$. Using this, we can solve for OX:

$$OX^3 = OX \cdot \frac{OY}{OX} \cdot \frac{k}{OY} = k.$$

We get that OX is $\sqrt[3]{k}$. Therefore, constructing the Beloch square results in the construction of the cube root of any number.

6. Doubling Cube

In section 1, we proved that all straightedge-compass constructible points must lie in a field of degree 2^k over $\mathbb{Q}$ for some finite positive integer k. In order to double the volume of a cube with a straightedge and compass, $\sqrt[3]{2}$ must be constructable. Since $\sqrt[3]{2}$ is algebraic over $\mathbb{Q}$ and it's irreducible polynomial is $x^3 - 2 = 0$, $\sqrt[3]{2}$ lies in a field extension of 3 over $\mathbb{Q}$. Therefore, $\sqrt[3]{2}$ is not an element in the straightedge-constructable field. It is impossible to construct with a straightedge and compass.
However, we could use origami - in square ABCD, we attempt to construct $\sqrt[3]{2}$ of the side length.

(1) Construct the midpoint J of side BC.
(2) Construct midpoint K of side CD
(3) Find the intersection L of lines AC and BK
(4) Construct a line MN parallel to line BC through L

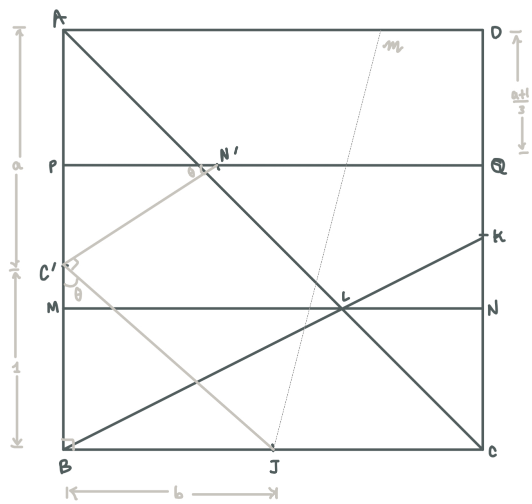


FIGURE 5. Doubling the Cube

(5) Construct a line PQ parallel to line MN halfway between MN and AD.
We have now divided side AB and DC into thirds with points M and P and N and Q, respectively.
(6) Use axiom 6 to create for R, placing C on AB at C' and N on PQ at N'
(7) AC' is $\sqrt[3]{2}$

Proof. Let us denote the length of the side of the square $a+1$. We aim to show that $a = \sqrt[3]{2}$. After step 6, we label BC' of length 1 and BJ of length b. We know that $JC' = JC = a+1-b$.

We solve for the value of PC'. This is $\frac{2}{3}(a+1) - 1 = \frac{2a-1}{3}$. Additionally, we can solve for the value of b using the pythagoran theorem on triangle $C'BJ$ to get $b = \frac{2a+a^2}{2(a+1)}$.

We have that angle $JC'N'$ is a right angle as it is just a projection of, and therefore congruent to the triangle JNC. From this, we can angle chase in triangles $BC'J$ and $C'PN'$ to get that $BC'J \sim C'PN'$ by AA.

Using the similarity, we have:

$$\frac{b}{a+1-b} = \frac{\frac{2a-1}{3}}{\frac{a+1}{3}}.$$

Once we substitute for b, we get:

$$\frac{a^2+2a}{a^2+2a+2} = \frac{2a-1}{a+1}.$$

Once we simplify and solve for a, we get $a^3 = 2$ so $a = \sqrt[3]{2}$. □

7. Trisecting the angle

Trisecting an angle is known to be impossible using straightedge and compass constructions. We can prove this by taking a specific angle and prove that that angle cannot be trisected.

Proposition 7.1. *It is impossible to trisect a* $60°$ *angle with a straightedge and compass*

Proof. We begin by considering the triple angle formula:

$$cos\theta = 4cos^3\frac{\theta}{3} - 3cos\frac{\theta}{3}$$

Let us form a polynomial in terms of $x = cos(20°)$, where $\theta = 60°$. We can simplify the equation after plugging in $cos(\theta)$ and x values to:

$$8x^3 - 6x - 1 = 0.$$

This equation can be simplified by substituting $2x$ for x because as as long as x is constructible, $2x$ will also be. We get:

$$x^3 - 3x - 1 = 0.$$

The roots of this polynomial are not straightedge- compass constructible. □

Now, we trisect the angle with origami.

Proposition 7.2. *Any angle can be trisected with origami.*

Proof. We begin with angle PQR. We aim to trisect angle PQR into 3 equal angles.

(1) Allow line p to be the perpendicular to QR at point Q.
(2) Let the foot of any perpendicular q to p be A.
(3) Let the foot of a perpendicular r to p, B be a point equidistant from A and Q.
(4) We construct fold m placing A onto PQ at A' and Q onto line r at Q'
(5) Let point B' be the image of B reflected across fold m.

We want to prove that PQB', $B'QQ'$, and $Q'QR$ equally trisect angle PQR. Let us begin by defining a few more points. Let N be the intersection of AQ' and $A'Q$, let S be the intersection of QQ' and the fold m, let V the intersection of AA' and the fold m, let W be the intersection of line q and DQ', let X be the intersection of line q and DS and let T be the foot of fold m (on QR).

Additionally, let us call $A'QB'$ as γ, $B'QQ'$ as δ, $A'QD$ as α, and DQM as β.

Now, we know that $\triangle DAX \sim \triangle DSQ$ by AA. Therefore, angle DXA is DQS, which is $\delta+\gamma+\alpha$. We know that, by reflection, angle $Q'DS$ is β, so angle $DB'A$ is $\alpha + \gamma + \delta - \beta$. We also know, by reflection, that angle $DQ'A = \alpha$, angle $AQ'B$ is γ, and $BQ'Q$ is δ. Therefore, angle $DQ'B$ is $\alpha+\gamma$. We also know that triangle DWA is similar to triangle $DQ'B$ by AA (lines r and q are parallel). Therefore, angle $DWA = DQ'B$, giving $\alpha + \gamma = \alpha + \gamma + \delta - \beta$. This gives us: $\delta = \beta$.

We also have that $\triangle NSQ \cong \triangle NSQ'$ by SAS because, by reflection, DS, or fold m, is the perpendicular bisector of QQ'. This gives us $AQ'B$ is γ and $BQ'Q$ is δ, or β. We know that $ABQ' \cong QBQ'$ by SAS, so angle $AQ'B$ is equal to angle $BQ'Q$, yielding $\gamma = \beta$.

Lastly, we have, by triangle DMQ, that angle $DMQ = 90 - \beta$. We also know that angle QSM is a right angle because DM, or fold m is the perpendicular bisector of QQ'. Therefore, angle $DMQ = SMQ = 90 - \beta$. This gives us, in triangle QSM, that angle $SQM = \beta$.

Therefore, $\beta = A'QB' = B'QQ' = Q'QR$. □

8. Squaring Circle

The objective of squaring the circle is, given a circle of area πr^2, construct a square of the same area.

In order to square the circle, $\sqrt{\pi}$ must be constructed. If $\sqrt{\pi}$ is constructible, π would also be constructible. It was first showed by Pierre Wantzel in 1837 that all straightedge and compass constructible lengths must be a solution to a polynomial with rational coefficients. In other words, all straightedge compass constructible lengths must be algebraic. In 1882, Ferdinand von Lindenman proved that π is transcendental[2] , yielding the impossibility of this construction.

Similarly, we cannot construct π with origami. Since π is transcendental, we know that $\mathbb{Q}(\pi)/\mathbb{Q}$ is not algebraic and $[\mathbb{Q}(\pi) : \mathbb{Q}]$ is infinite, which cannot be expressed in the form 2^k for some integer k as described in section 1. Therefore, the squaring of the circle is also impossible to do with origami.

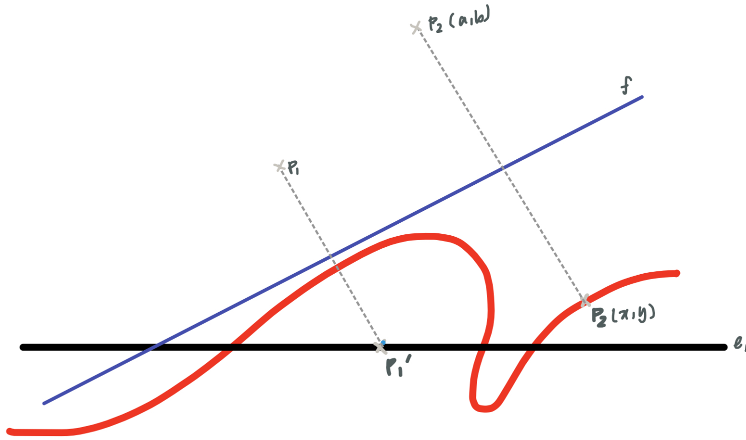


FIGURE 6. Cubic Example

9. Solving a Cubic

Proposition 9.1. *Performing the Beloch fold on 2 points P_1 and P_2 and 2 lines l_1 and l_2 is equivalent to solving a cubic.*

Proof. Define points $P_1(0,1)$ and $P_2(a,b)$, and line l_1 as the horizontal line $y=-1$. We observe the point where P_2 is folded to when we fold P_1 with different creases, P_2', or (x,y). We established in section 5 that the crease line, f, that we constructed is tangent to the parabola formed with focus P_1 and directrix l_1. Additionally, f is tangent to the parabola formed with focus P_2 and directrix l_2.

Recall from section 5: The equation of the crease f is $y=\frac{t}{2}x-\frac{t^2}{4}$. f is also the perpendicular bisector of P_2P_2'. Calculating the slope and midpoint gives us: $\frac{y-b}{x-a}$ and $(\frac{a+x}{2},\frac{b+y}{2})$. We know that P_2P_2' is perpendicular to f, giving us the slope of P_2P_2' as $\frac{-2}{t}$. We equate the two slopes to get equation 1:

$$\frac{-2}{t}=\frac{y-b}{x-a} \tag{1}$$

Additionally, we observe that the midpoint of P_2P_2' lies on the crease to get equation 2:

$$\frac{b+y}{2}=\frac{t}{2}\cdot\frac{a+x}{2}-\frac{t^2}{4} \tag{2}$$

If we substitute (1) into (2), we get a cubic.

$$(y+b)(y-b)^2=-(x^2-a^2)(y-b)-2(x-a)^2 \tag{3}$$

Folding P_2 onto line l_2 creates another intersection with the cubic curve. Solving the cubic above at a certain point is equivalent to finding the intersection point. □

10. Introducing Another Fold

There are two commonly used folds: the multifold and the single fold. The single fold is what we have been working with thus far. Using single folds, we can derive 8 axioms from which we derive equations and solve (sections 2-3).

Definition 10.1 (Multifold)**.** An n-fold multifold defines n (greater than 1) simultaneous folds on a plane with finite roots.

The multifold allows for 203 axioms. This enables us to solve a multitude of problems that we couldn't sole with single fold origami. For example, a useful multifold axiom is AL4a6ab.

Algorithm AL4a6ab is comprised of two folds based on 2 points, P and Q, and 3 lines, l, m, and n that defines 2 creases ξ and χ. These creases are determined in 3 alignments:

(1) $\chi(P) \in m$
(2) $\chi(l) = \xi$
(3) $\xi(Q) \in n$

In other words, given 2 points, P and Q, and 3 lines, l, m, and n, fold χ places point P onto line l and ξ places point Q onto line m and line n onto fold χ. As described in section 1, some fold χ

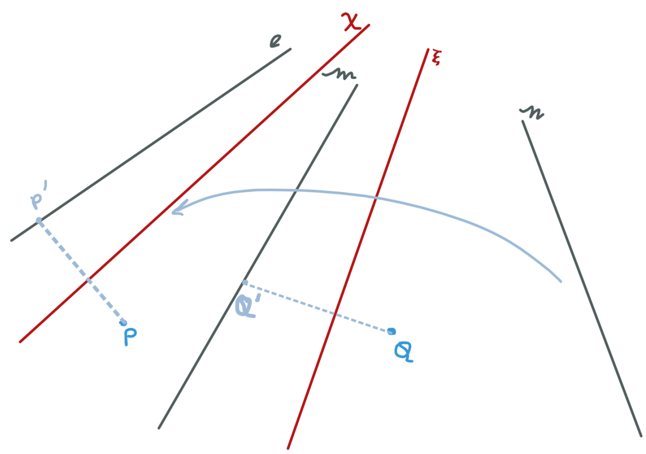


Figure 7. AL4a6ab

determined by alignment $\chi(P) \in m$ is a tangent to a parabola, C_a with focus P and directrix m.

In his paper, Nishimura[6] synthesizes the fold: The algorithm AL4a6ab is the same as the intersection of the line χ and the locus

$$C := \{\chi(l)(Q) \mid \chi \text{ is a tangential line of } C_a\}.$$

11. Solving a Quintic

A quintic is solvable by radicals if its Galois group is solvable.[3] A group F is said to be solvable if it has a finite series of subgroups

$$1 = G_0 \subseteq G_1 \subseteq \cdots \subseteq G_n = G$$

such that for every i, $1 \leq i \leq n$, g_{n-1} is a subgroup of G and the factor group G_n/G_{n-1} an abelian group. In 1867, Eduard Lill designed an algorithm that presented a geometric solution to any polynomial.

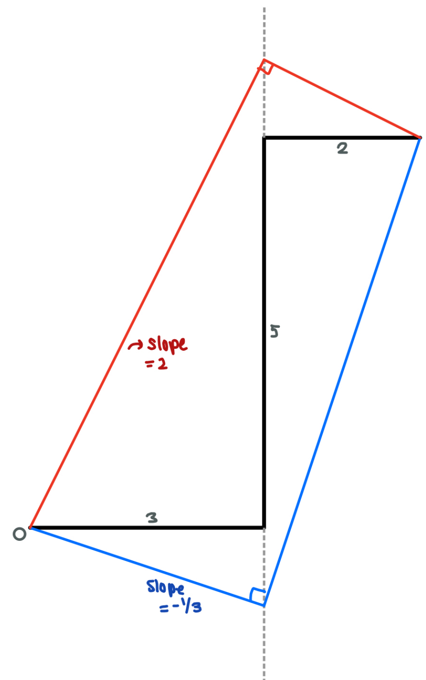
Say we have a polynomial of form $a_n x^n + a_{n-1} x^{n-1} + a_{n-2} x^{n-2} + \cdots + + a_2 x^2 + a_1 x + a_0$. We construct a path consisting of the unit lengths of each coefficient $a_n, a_{n-1}, a_{n-2}, \ldots, a_2, a_1$, and a_0, beginning at the origin with a_n along the x-axis. After every segment, we take a 90° counterclockwise turn.

We construct a second path as follows:

(1) The path begins at the origin at some angle.

(2) Whenever the second path encounters the first path or any extensions of the first path, it reflects at 90°.

For example, we have polynomial $3x^2 + 5x - 2$. We have the first path drawn in black in the figure below. We have 2 options for the second path, depicted in blue and red. The negative slopes of the first segments of the blue and red paths, $\frac{1}{3}$ and -2, are the solutions to the polynomial.



Proposition 11.1. *The solutions to a polynomial* $a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 = 0$ *are the negative slope of the first segment of the second path.*

Proof. The sides of the second path consist of the hypotenuses to a set of similar right triangles with one side on the first path.
Let us define y_k as the length opposite the ricocheting angle. For example:
We know that the following set of equations is true:

$$y_n = \text{slope} \cdot a_n = -xa_n$$

$$y_{n-1} = \text{slope} \cdot (a_{n-1} - (-xa_n) = -x(a_{n-1} + xa_n)$$

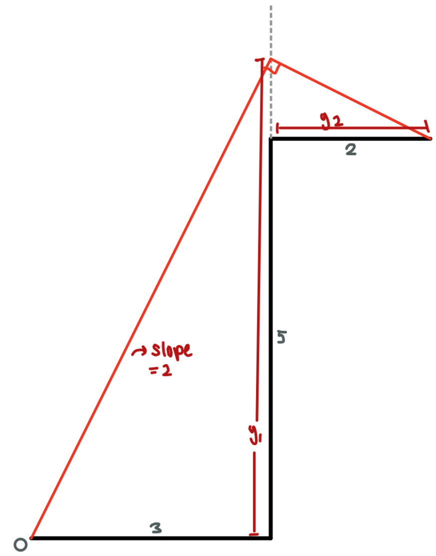
$$y_{n-2} = \text{slope} \cdot (a_{n-2} - (-x(a_{n-1} - (-xa_n)) = -x(a_{n-2} + x(a_{n-1} + xa_n))$$

$$\vdots$$

$$y_1 = -x(a_1 + x(a_2 + \ldots. + x(a_{n-2} + x(a_{n-1} + xa_n))\ldots)) \tag{4}$$

However, we know that $y_1 = a_0$. Equating (4) and a_0 simplifies to $f(x) = 0$.
Lill's method works on a polynomial based off of Horner's method. The polynomial $a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ has distances of $a_n x$, $(a_{n-1} + a_n)x$, $((a_{n-1} + a_n)x + a_{n-2})x, \ldots$ between the vertices of the polynomial and root paths. Lill's method is simply a visual construction of synthetic division by a linear root.

□



As an extension of this, we have:

Proposition 11.2. *A polynomial of degree n can be solved with $n-2$ simultaneous folds.*

Proof. We can use Lill's method to reason this. A polynomial with degree n, when drawn out using Lill's method, would be composed of n segments and $n-1$ folds. We can apply a certain algorithm to specific segments and turns:
Assume we use a multitude of two fold algorithms (AL1-AL10) for all the $n-3$ intermediate turns. This would give us a total of $2(n-3)+2 = 2(n-2)$ equations.

Now, assume we use the two fold algorithm AL6[4] for the first and last segment and $n-2$ fold axiom for the $n-2$ intermediate segments.
This would give us a total of $2(n-2)$ equations, equivalent to the number of equations from the turns.

Therefore, the same amount of equations used to solve the polynomial formed from the turns is formed from $n-2$ simultaneous folds. □

From the section above, we can say that all quintics can be solved with 3-fold origami. However, Alper and Lang[4] attempted to do better and found a method to solve quintics with 2 fold origami:

Proposition 11.3. *The quintics are solvable with axiom AL4a6ab.*

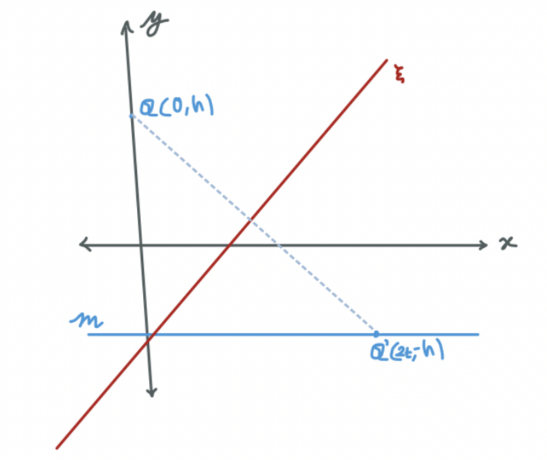
Proof. As described in section 10, algorithm AL4a6ab is a two-fold algorithm, composed of 2 folds done simultaneously:

Fold 1: On a coordinate plane $Q(0,h)$ and $m: y=-h$, Q is folded over ξ onto line m to point $Q'(2t,-h)$.

We describe fold ξ in vector form in terms of the normal vector $\overrightarrow{n}$: $\overline{QQ'} = < 2t, -2h >$, point $x(x,y)$, and point x_0 on line ξ as $\overrightarrow{n} \cdot x = \overrightarrow{n} \cdot x_0$. We can choose x_0 to be $(t,0)$, the midpoint of $\overline{QQ'}$. Substituting and evaluating, we get

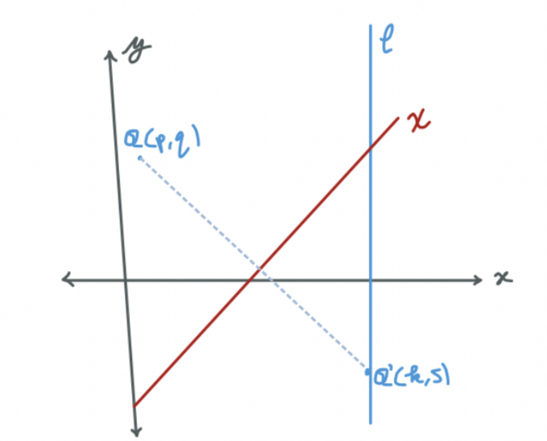
$$t^2 = tx - hy. \tag{5}$$

Fold 2: On a coordinate plane $P(p,q)$ and $l: x=k$, P is folded over χ onto line l to point $P'(k,s)$. Again, we describe fold χ in vector form in terms of



the normal vector $\overrightarrow{n} : \overline{PP'} =< k-p, s-q >$, point $x(x,y)$, and point x_0 on line χ as $\overrightarrow{n} \cdot x = \overrightarrow{n} \cdot x_0$. Substituting $\overrightarrow{n}$ and x_0 as $(\frac{s+q}{2}, \frac{k+p}{2})$, we have the equation for χ as:

$$\frac{s^2-q^2}{2} + \frac{k^2-p^2}{2} = (k-p)x + (s-q)y \tag{6}$$



Define line n as the imprint of χ folded over ξ. We set n as $ax + by = c$.

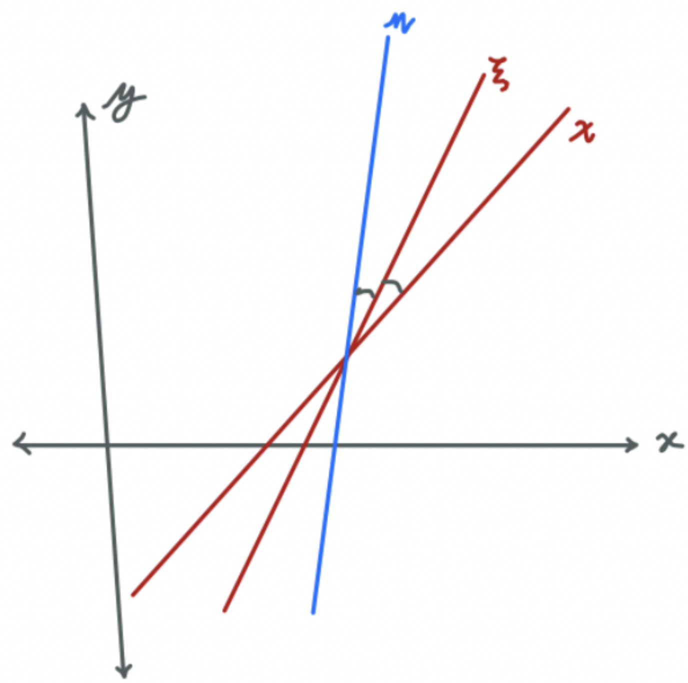
We now have 2 cases: ξ and n intersect or ξ and n do not intersect.

Let us first take the case in which they do intersect: We solve for the intersection point of ξ and n using the equations of both to get the intersection as

$$\left(\frac{bt^2+ch}{bt+ah}, -\frac{t(at-c)}{bt+ah}\right) \tag{7}$$

We also know that the intersection is on χ, so we plug in the point in the equation of χ to get:

$$(k-p)\frac{bt^2+ch}{bt+ah} - (s-q)\frac{t(at-c)}{bt+ah} = \frac{s^2-q^2}{2} + \frac{k^2-p^2}{2} \tag{8}$$



Additionally, fold χ bisects the angle θ between ξ and n. We then get:

$$cos\frac{\theta}{2} = \frac{|\overrightarrow{n_\xi} \cdot \overrightarrow{n_\chi}|}{||\overrightarrow{n_\xi}||\ ||\overrightarrow{n_\chi}||} = \frac{|\overrightarrow{n_\xi} \cdot \overrightarrow{n_n}|}{||\overrightarrow{n_\xi}||\ ||\overrightarrow{n_n}||}.$$

When we substitute n_ξ, n_χ, and n_n for $(t, -h), (k-p, s-q)$, and (a, b), respectively, to get:

$$\frac{|t(k-p) - h(s-q)|}{\sqrt{(k-p)^2 + (s-q)^2}} = \frac{|at - bh|}{\sqrt{a^2 - b^2}}.$$

This equation can be substituted along with equation (8) to get a quintic:

$$t^5 + \alpha t^4 + \beta t^3 + \gamma t^2 + \delta t + \epsilon = 0.$$

When we solve for the coefficients, we get:

(9) $$\alpha = (-k - p + 2bq - b^2k + b^2p - 12bc - 2c)/4$$

(10) $$\beta = h(q + 2bp - b^2q + bc - h + 2b^2h)$$

(11) $$\gamma = h^2(3p - k - 6bq - b^2k - 3b^2p + 2bh)/2$$

(12) $$\delta = -h^3(q + 2bp - b^2q - bc)$$

(13) $$\epsilon = h^4(-k - p + 2bq - b^2k + b^2p + 2c)/4.$$

Now, we solve for P, Q, l, m, and n. Using equations (9) and (13), we get:

(14) $$\epsilon - h^4\alpha = h^4(c + 3bh)$$

Using equations (10) and (12), we get:

(15) $$b^2\beta + \delta = h^3(2bc - 2b^2h - h)$$

Using equations (14) and (15), we have:

$$b = \frac{\epsilon - h^4\alpha \pm \sqrt{(\epsilon - h^4\alpha)^2 - 4h^6(h^4 + h^2\beta + \delta)}}{4h^5}$$

$$c = \frac{\epsilon - h^4\alpha \pm 3\sqrt{(\epsilon - h^4\alpha)^2 - 4h^6(h^4 + h^2\beta + \delta)}}{4h^4}.$$

From the set of equations (9-13), we can substitute the values of b and c to solve for k, p, and q:

$$k = -\frac{17bh^3 + 3h^2(c + 2\alpha) - \gamma}{2h^2(b^2 + 1)}$$

$$p = \frac{-bh^3(b^2 - 3) + h^2((2a - 3c)b^2 - c^2 - 2\alpha) + 4bh\beta + (1 - b^2)\gamma}{2h^2(b^2 + 1)^2}$$

$$q = \frac{h^3(2b^4 + 4b^2 + 1) + bh^2(b^c + 2\alpha) + \beta h(1 - b^2) - b\gamma}{h^2(b^2 + 1)^2}$$

Now, we take the case in which ξ and n do not intersect. We immediately see that some adjustments must be made:

(1) The normal vectors to ξ and n ($< t, -h >$ and $< a, b >$, respectively) are parallel. Therefore, we have:

$$bt + ah = 0. \tag{16}$$

(2) Since fold ξ places fold χ upon line n, all 3 are parallel to each other. Therefore, we get another equation from the normals of the 3 vectors:

$$b(k - p) - a(s - q) = 0. \tag{17}$$

(3) The distance between ξ and χ is equal to the distance between ξ and n. The distance between 2 parallel lines is:

$$\left| \frac{|c_1 - c_2|}{\sqrt{a^2 + b^2}} \right|,$$

where c_1 and c_2 are the y-intercepts of the two lines and a and b are the coefficients of x and y, respectively.

Substituting equation (16), we can alter the equation of ξ to $ax - by = -a^2h/b$ and solve for the distance between ξ and n:

$$d = \frac{|ah^2/b + c|}{\sqrt{a^2 + b^2}}.$$

Similarly, we can alter the original equation of χ using equation (17) to get $ax + by = (a^2(k + p) + 2abq + b^2(k - p))/2a$ and solve for the distance between ξ and χ:

$$d = \frac{|ah^2/b + (a^2(k + p) + 2abq + b^2(k - p))/2a|}{\sqrt{a^2 + b^2}}.$$

Setting the two distances equal to each other gives:

$$\frac{a^2(k + p) + 2abq + b^2(k - p)}{2a} = c \tag{18}$$

and

$$4a^3h + a^2b(k + p) + 2ab(bq + c) + b^3(k - p) = 0. \tag{19}$$

We now compare the 2 new conditions:

(1) Equation (18) compares the altered equation of χ and the original equation of n, $ax+by = c$, implying that n is equal to χ. This can be disregarded.

(2) Equation (19) can be shown using equations (16) and equations (9-13), so this condition, too, can be disregarded.

This shows that case 2, the case when ξ and n are parallel, is entirely contained within the first case. The solution found in case 1 stands regardless of the intersection of ξ and n.

So, we still have the same solutions as in case 1. We are able to solve a quintic with two-fold origami. □

References

[1] Thomas C Hull *Origametry: Mathematical Methods in Paper Folding*. 2021.

[2] Randy K Schwartz *Pi is Transcendental: Von Lindemann's Proof Made Accessible to Today's Undergraduates* . 2006.

[3] Yunye Jiang *Galois Theory and the Quintic Equation*. 2018.

[4] R. Alperin and R. Lang *One-, two, and multi-fold origami axioms, Origami 4*. 2006.

[5] H. Lee *Origami Constructible Numbers*. 2017.

[6] Y. Nishimura *Solving quintic equations by two-fold origami, Forum Mathematicum, 27* 2015.

The author would like to credit the following papers. Although not directly referenced, they played an immense part in the content of the paper:

1. James King: Origami Constructible Numbers
2. Moti Ben-Ari: The Mathematics of Origami
3. Spencer Chan: Compass and Straightedge Applications of Field Theory
4. Tanner Struck: Field Theory for Compass and Straightedge Impossibility Proofs