The Black-Scholes Model

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Introduction

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- Black and Merton received a Nobel Memorial Prize in Economicomic Sciences for deriving this model in 1997. Unfortunately, Black was unable to receive the award due to passing away 2 years prior.
- Today, the model is still used by investors today, in addition to variants of the model that people have derived for more exotic types of options.

Important Economic Terminology

Definition

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A *European option* is simply an option that can be exercised only at the expiration of the option, which is specified in the contract.

Important Economic Terminology (Cont.)

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Now that we are equipped with the economic knowledge required to understand this model, we can begin exploring it.

The Black-Scholes Model

$$C = S_0 N\left(\frac{rT + \frac{v^2 T}{2} + \ln\left(\frac{S_0}{K}\right)}{v\sqrt{T}}\right) - Ke^{-rT} N\left(\frac{rT + \frac{v^2 T}{2} + \ln\left(\frac{S_0}{K}\right)}{v\sqrt{T}}\right)$$

In this model, C represents the value of the call option, S_0 is the initial price of the stock, $N(x)$ represents the cumulative distribution function of a standard normal variable, r is the risk-free interest rate, K is the strike price of the option, T is the amount of time until the option expires, and v is the annual volatility of the stock price.

Assumptions

For this proof, we will assume we are only dealing with European options, meaning they can only be exercised at the expiration date.

No dividends are paid out throughout the duration in which the option is active.

Both the risk-free rate and volatility of the given asset are known and are constant.

The price of a stock is log-normally distributed.

Buying an option does not include any additional transaction costs.

Assumptions (Cont.)

Markets are random, so movements cannot be predicted.

There are no arbitrage possibilities.

Trading of the asset can take place at any time; continuously.

This is minor, however still important to note. We can buy or sell any number of the asset, which includes non-integer numbers.

With this, we can finally dive into the math related to proving the model.

Proof Preparation (Cont.)

The first thing we will be doing is going over some probability ideas that will be very essential for our proof.

Definition

The *cumulative distribution function*, F, of the random variable X is defined for all real numbers b by:

$$F(b) = \mathbf{P}\{X \leq b\}$$

We can say that X acknowledges a *probability density function* if:

$$\mathbf{P}\{X \leq b\} = F(b) = \int_{-\infty}^{b} f(x)dx$$

Proof Preparation (Cont.)

Definition

We say X is a *normal random variable* with parameters μ and $\sigma^2 > 0$ if we say that $-\infty < x < \infty$ and the density of X is

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{\frac{-(x-\mu)^2}{2\sigma^2}}$$

This means we can define a cumulative distribution function of a normal random variable with a mean of 0 and a variance of 1 by:

$$N(x) = \frac{1}{2\pi} \int_{-\infty}^{x} e^{\frac{-y^2}{2}}$$

Proof Preparation (Cont.)

Proof Preparation (Cont.)

After defining random normal variables and cumulative distribution functions, we must take a look at expected value as it is the main idea that our proof is centered around.

Definition

Expected Value is a way to estimate the final result. Expected value is mainly found using weighted averages. In this case, we will be defining an integral for our specific case of finding the expected value of an option, namely that the expected value of a given continuous random variable X having a probability density function f(x) can be given by:

$$\mathbf{E}[X] = \int_{-\infty}^{\infty} x f(x) dx$$

Proof Preparation (Cont.)

Definition

We define a *universe* as a class that contains all the entities one needs to consider for a given situation.
The idea behind a universe is it allows us to isolate conditions that we need to have for a theorem to be true. In this case, we will be assuming that the risk-free interest rate is available to us at all times.

Construction

We construct a "forward price" of a stock as the current price plus the return which will exactly offset the cost and risk of holding the given asset over the period of time t. The cost here is the risk-free interest lost, so we have: $S_0 e^{rt}$

Proof Preparation (Cont.)

Definition

We call a universe *risk neutral* if at the expected value of the asset A and time period t, C(A,0) at t = 0 is the expected value of the asset at time t deducted to its present value, again utilizing the risk free rate r.

$$C(A, 0) = e^{-rt}\mathbf{E}[C(A, t)]$$

Now that we've finished with all the probability theory needed, we will be looking at some values and equations that will be important for our proof later.

Proof Preparation (Cont.)

These are just equations and values I will be stating, but feel free to contact me or read my paper for the proof of them.

Lemma

$$\sigma = v^2 t$$

Lemma

$$\mu = (r - \frac{v^2}{2})t$$

Equation

$$\mathbf{E}[S_t] = \frac{S_0 e^{\mu+\sigma/2}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{\frac{-x^2}{2}}$$
$$= S_0 e^{\mu+\sigma/2}$$

Proof

We have that $C(S,t) = max(S_T - K, 0)$

$$
\begin{aligned}
C(S,0) &= e^{-rT}\mathbf{E}[C(S,T)] \\
&= e^{-rT}\mathbf{E}[max(S_T - K, 0)] \\
&= e^{-rT}\int_K^\infty \frac{1}{\sqrt{2\pi T}vx}(x-K)e^{\frac{-(ln\frac{x}{S_0-\mu^2})^2}{2v^2T}}dx \\
&= e^{-rT}\int_K^\infty \frac{1}{\sqrt{2\pi T}v}e^{\frac{-(ln\frac{x}{S_0-\mu^2})^2}{2v^2T}}dx \\
&\qquad - e^{-rT}\int_K^\infty \frac{1}{\sqrt{2\pi T}vx}Ke^{\frac{-(ln\frac{x}{S_0-\mu^2})^2}{2v^2T}}dx
\end{aligned}
$$

Proof (Cont.)

We observe that the first integral is identical to the one we had in our Lemma. Therefore, the first term simplifies to:

$$e^{-rT} S_0 e^{\mu + \frac{v^2 T}{2}} \int_A^{\infty} \frac{1}{\sqrt{2\pi}} e^{\frac{-y^2}{2}} dy$$

where

$$A = \frac{ln \frac{K}{S_0} - \mu - v^2 T}{v\sqrt{T}}$$

Note that this is the same integral we had when defining a risk-free universe ($C(A, 0) = e^{-rt}\mathbf{E}[C(A, t)]$) except now the lower limit is A.

Proof (Cont.)

We proceed to use the equation for $\mathbf{E}[S_t]$ to find the value of μ, and then realize that the resulting integral represents the cumulative distribution function for the standard normal variable, so we have that:

$$S_0\left(1 - N\left(\frac{ln\frac{K}{S_0} - rT - \frac{v^2T}{2}}{v\sqrt{T}}\right)\right) = S_0 N\left(-\frac{ln\frac{K}{S_0} - rT - \frac{v^2T}{2}}{v\sqrt{T}}\right)$$
$$= S_0 N\left(\frac{ln\frac{K}{S_0} + rT + \frac{v^2T}{2}}{v\sqrt{T}}\right)$$

and with that we have the first term of the Black-Scholes Model.

Proof (Cont.)

For our sEconomicd term, we again look to simplify. To do this, we first start by letting

$$z = \frac{ln\frac{x}{S_0} - \mu}{v\sqrt{T}}$$

This allows us to find the derivative of z to be:

$$dz = \frac{dx}{xv\sqrt{T}}$$

All we have to do now is substitute and simplify, and we will have found the sEconomicd term of the model.

Proof (Cont.)

$$
\begin{aligned}
-e^{-rT}\int_K^{\infty}\frac{1}{\sqrt{2\pi T}vx}Ke^{\frac{-(ln\frac{x}{S_0-\mu^2})^2}{2v^2T}}dx &= -e^{-rT}\int_{A+v\sqrt{T}}^{\infty}\frac{1}{\sqrt{2\pi}}(x-K)e^{\frac{-z^2}{2}}dz \\
&= -e^{-rT}K\Bigg(1-N\Bigg(A+v\sqrt{T}\Bigg)\Bigg) \\
&= -Ke^{-rT}N\Bigg(-A-v\sqrt{T}\Bigg) \\
&= Ke^{\text{-rT}}N\Bigg(\frac{rT+\frac{v^2T}{2}+\ln\left(\frac{S_0}{K}\right)}{v\sqrt{T}}\Bigg)
\end{aligned}
$$

With that, we have found the sEconomicd term of the model and have completed our proof.

Extensions

Extensions

- Dividends

Extensions

- Dividends
- Perpetuity

Thank You

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