THE BLACK-SCHOLES MODEL :)

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ABSTRACT. This paper will derive the Black-Scholes pricing model of a European option in 2 ways. We will assume that the price of a stock is log-normally distributed. We will also make use of Ito's Lemma to justify the assumptions made in our first proof. In addition to this, we will prove put-call parity to price European put options, and finally extend the original model derived by Black and Scholes to value a couple other different options.

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1. INTRODUCTION

The Black-Scholes model, aka the Black-Scholes-Merton (BSM) Model, was developed by Fischer Black, Robert Merton and Myron Scholes in 1973. Robert and Myron would go on to win the Nobel Peace Prize for developing this theorem, and their work heavily influences the techniques and pricing models used in our finance world today. This paper will act as an introduction to this model, a way to explore the countless possibilities of this formula, and will also highlight two methods to derive this fascinating model. We will start with the more intuitive and less rigorous proof, and then follow that up with the more formal and concrete proof. Both of these will end up leading to the same end result, namely that the "fair value" of an European call option is

$$C = S_0 N\left(\frac{rT + \frac{v^2 T}{2} + \ln\left(\frac{S_0}{K}\right)}{v\sqrt{T}}\right) - Ke^{-rT} N\left(\frac{rT + \frac{v^2 T}{2} + \ln\left(\frac{S_0}{K}\right)}{v\sqrt{T}}\right)$$

where S_0 is the initial price of the stock, $N(x)$ represents the cumulative distribution function of a standard normal variable, r is the risk-free interest rate, K is the strike price of the option, T is the amount of time until the option expires, and v is the annual volatility of the stock price.

Date: July 11, 2022.

2. Economic Background Knowledge

We begin with a review of some basic economic terminology needed for understanding and deriving this equation.

Definition 2.1. A *stock* is something that represents your ownership right to a small part of a public company. A stock can go up if there is high demand for it, and it will go down if there is low demand for a set amount of shares. A stock can also go down by diluting the stock, or increasing the amount of outstanding shares in the company.

There are two main types of options that most people purchase, call and put options that we will be introducing in the following definitions. In addition to that, we will also be talking about how European options work, as it is what the original Black-Scholes Model was made for.

Definition 2.2. A *call option* is a contract where the holder of the option has the right (not the obligation) to buy an asset at a certain time in the future for a specific price, called the strike price.

Definition 2.3. A *put option* is a contract where the holder of the option has the right (not the obligation) to buy an asset at a certain time in the future for a specific price, called the strike price.

Definition 2.4. A *European option* is simply an option that can be exercised only at the expiration of the option, which is specified in the contract.

Definition 2.5. The *risk-free rate* is the rate of return of an investment with no risk of loss. Commonly, this is found using T-Bills, as they are fully backed by the government. To calculate the risk-free rate, subtract the inflation rate from the yield of the Treasury bond matching your investment duration.

Two things that potential option and also stock buyers consider when buying a stock is if it pays dividends (in addition to if so, the size of the dividend), and also how volatile or risky a certain stock is.

Definition 2.6. A *dividend* is a portion of profits distributed to shareholders by the corporation. This distribution typically happens quarterly.

Definition 2.7. *Volatility* represents the frequency and magnitude of price movements in a given investment. The more frequent/large these price movements are, the "more" volatile an investment is.

Definition 2.8. *Arbitrage* is an investment strategy that is utilized to take advantage of a price difference and turn it into profit. This is done by buying and selling the same asset simultaneously in different markets.

With this knowledge, we can now proceed onto our first derivation of this model.

3. Expected Value Derivation

The first method we will employ to derive the formula uses expected value to find the "fair value" of an call option, a different method than the one originally employed by Black and Scholes. We choose to employ this method first due to its intuitiveness. We will then follow this up with a more rigorous proof using the method that was used back in 1973 when this model was derived.

3.1. **Assumptions.** There are a few assumptions and terms related to probability theory that we must go over before we proceed with the proof. We will start with all the assumptions we must make in this derivation, followed by the probability theory terminology that will be required to prove this theorem. These assumptions are vital to talk about as it shows the flaws of the Black-Scholes Model and shows that it cannot account for everything.

Assumption 3.1. For this proof, we will assume we are only dealing with European options, meaning they can only be exercised at the expiration date.

Assumption 3.2. No dividends are paid out throughout the duration in which the option is active.

Assumption 3.3. Markets are random, so movements cannot be predicted.

Assumption 3.4. Buying an option does not include any additional transaction costs.

Assumption 3.5. Both the risk-free rate and volatility of the given asset are known and are constant.

Assumption 3.6. There are no arbitrage possibilities. Arbitrage is the simultaneous purchase and sale of the same asset in different markets in order to profit from tiny differences in the asset's listed price. Typically, they buy in huge volumes, as even with really low margins, the high volumes of purchase make it worthwhile. As an example, if a stock is worth a dollar on one exchange and two dollars on another, someone would utilize arbitrage by purchasing the one dollar stock and selling the stock for two dollars at the other exchange at the same time.

Assumption 3.7. Trading of the asset can take place at any time; continuously.

Assumption 3.8. The price of a stock is log-normally distributed.

Assumption 3.9. This is minor, however still important to note. We can buy or sell any number of the asset, which includes non-integer numbers.

3.2. **Probability Terminology.** Moving onto the probability theory side of things, we start by defining expected value, as it is the main idea that our proof is centered around.

The first thing we will be doing is going over some probability ideas that will be very essential for our proof.

Definition 3.10. The *cumulative distribution function*, F, of the random variable X is defined for all real numbers b by:

$$F(b) = \mathbf{P}\{X \leq b\}$$

We can say that X acknowledges a *probability density function* if:

$$\mathbf{P}\{X \leq b\} = F(b) = \int_{-\infty}^{b} f(x)dx$$

In this paper, we will be using a bold-faced P to represent a probability density function.

Definition 3.11. We say X is a *normal random variable* with parameters μ and $\sigma^2 > 0$ if we say that $-\infty < x < \infty$ and the density of X is

$$f(x) = \frac{1}{\sqrt{2\pi\sigma}} e^{\frac{-(x-\mu)^2}{2\sigma^2}}$$

This means we can define a cumulative distribution function of a normal random variable with a mean of 0 and a variance of 1 by:

$$N(x) = \frac{1}{2\pi} \int_{-\infty}^{x} e^{\frac{-y^2}{2}}$$

After defining random normal variables and cumulative distribution functions, we must take a look at expected value as it is the main idea that our proof is centered around.

Definition 3.12. *Expected Value* is a way to estimate the final result. Expected value is mainly found using weighted averages. In this case, we will be defining an integral for our specific case of finding the expected value of an option, namely that the expected value of a given continuous random variable X having a probability density function f(x) can be given by:

$$\mathbf{E}[X] = \int_{-\infty}^{\infty} x f(x) dx$$

Definition 3.13. The random variable X is *log-normally distributed* if there is a normally distributed variable Y, such that $x = e^Y$, or that $\ln X$ is normally distributed.

Definition 3.14. We define a *universe* as a class that contains all the entities one needs to consider for a given situation.

The idea behind a universe is it allows us to isolate conditions that we need to have for a theorem to be true. In this case, we will be assuming that the risk-free interest rate is available to us at all times. Using the idea of a universe will make certain simplifications in our argument accessible.

Definition 3.15. We call a universe *risk neutral* if at the expected value of the asset A and time period t, C(A,0) at t = 0 is the expected value of the asset at time t deducted to its present value, again utilizing the risk free rate r.

$$C(A, 0) = e^{-rt}\mathbf{E}[C(A, t)]$$

Construction 3.16. We construct a "forward price" of a stock as the current price plus the return which will exactly offset the cost and risk of holding the given asset over the period of time t. The cost here is the risk-free interest lost, so we have:

$$S_0 e^{rt}$$

Lemma 3.17. *Let S_0 be the initial value of the stock price, S_t be the price at time t, and let v represent the annual volatility in percent change of the stock price (i.e. the standard deviation of the percent change over the course of a year.) Finally, assume S_t to be a log-normally distributed random variable (i.e. $\ln \frac{S_t}{S_0}$) is normally distributed with a mean of μ and variance σ, and finally let the mean of the log-normal distribution be at the forward price of the stock, as we previously defined it. Then, we have $\mu = \mu(t)$, $\sigma = \sigma(t)$, and*

$$\sigma = v^2 t \tag{3.1}$$

$$\mu = (r - \frac{v^2}{2})t \tag{3.2}$$

It may not seem clear how we will use this or how this is even remotely related to our proof, but it is very useful in our derivation. It allows us to simplify very complex expressions easily and gives a nice result while allowing up to compute the drift of the stock, or μ, in terms of the risk-free interest rate, annual volatility, and time.

Proof. To prove equation 3.1, we use induction. We observe that after one year, $\ln \frac{S_1}{S_0}$ has variance $(v^2)1$. We say that after $t-1$ years, $\ln \frac{S_{t-1}}{S_0}$ has variance $(v^2)(t-1)$, and what follows is that after t years, we have

$$\begin{aligned}\ln \frac{S_t}{S_0} &= \ln \frac{S_t - 1S_t}{S_0 S_{t-1}} \\ &= \ln \frac{S_{t-1}}{S_0} + \ln \frac{S_t}{S_{t-1}}\end{aligned}$$

and therefore has variance $(v^2)(t-1) + v^2 = v^2 t$

Equation 3.2 comes from the following:

$$\begin{aligned}F(a) &= \mathbf{P}\{S_t \leq a\} \\ &= \mathbf{P}\{S_0 e^{x_t} \leq a\} \\ &= \mathbf{P}\{x_t \leq \ln \frac{a}{S_0}\} \\ &= \frac{1}{\sqrt{2\sigma\pi}} \int_{-\infty}^{\ln \frac{a}{S_0}} e^{\frac{-(x_t - \mu)^2}{2\sigma}}\end{aligned}$$

We get the density function for S_t by differentiating the integral with respect to a to get:

$$f(x) = \frac{1}{\sqrt{2\sigma\pi x}} e^{\frac{-(\ln \frac{x}{S_0} - \mu)^2}{2\sigma}}$$

By assumption, we have that $\mathbf{E}[S_t] = S_0 e^{rt}$, so

$$\begin{aligned}\mathbf{E}[S_t] &= \int_0^\infty \frac{1}{\sqrt{2\sigma\pi x}} x e^{\frac{-(\ln \frac{x}{S_0} - \mu)^2}{2\sigma}} \\ &= \frac{1}{\sqrt{2\sigma\pi}} \int_0^\infty e^{\frac{-(\ln \frac{x}{S_0} - \mu)^2}{2\sigma}}\end{aligned}$$

Now we utilize u-substitution to solve this integral. Let $z = \frac{\ln \frac{x}{S_0} - \mu}{\sqrt{\sigma}}$, then $dz = \frac{dx}{x\sqrt{\sigma}}$, where $x = S_0 e^{z\sqrt{\sigma} + \mu}$, and what follows is that:

$$\begin{aligned}\mathbf{E}[S_t] &= \frac{S_0}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{\frac{-z^2}{2}} e^{z\sqrt{\sigma}+\mu} dz \\ &= \frac{S_0}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{\frac{-z^2}{2}+z\sqrt{\sigma}+\mu} dz \\ &= \frac{S_0}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{\frac{-(z-\sigma)^2}{2}+\sigma/2+\mu} dz \\ &= \frac{S_0 e^{\mu+\sigma/2}}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{\frac{-z^2}{2}+z\sqrt{\sigma}+\mu} dz\end{aligned}$$

Now the final step here is to let $x = z - \sqrt{\sigma}$, and that leads us to see that:

$$\begin{aligned}\mathbf{E}[S_t] &= \frac{S_0 e^{\mu+\sigma/2}}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{\frac{-x^2}{2}} \\ &= S_0 e^{\mu+\sigma/2}\end{aligned} \tag{3.3}$$

We observe that the last integral is equal to $\sqrt{2\pi}$. Since $\sigma = v^2 t$, $S_0 e^{\mu+\frac{v^2 t}{2}} = S_0 e^{rt}$ and $\mu = (r - \frac{v^2}{2})t$ ■

In this paper, expected value will always be given by a bold **E**, while a normal E will not and instead represent some variable or boundary.

Theorem 3.18. *(Black-Scholes) In a risk-neutral universe with an initial stock price S_0, at time t, and a log-normally distributed stock price S_t, the value C of a European call option at time $t = 0$ with strike price K, and expiration time T, and r being the risk-free rate is:*

$$C = S_0 N\left(\frac{rT + \frac{v^2 T}{2} + \ln\left(\frac{S_0}{K}\right)}{v\sqrt{T}}\right) - Ke^{-rT} N\left(\frac{rT + \frac{v^2 T}{2} + \ln\left(\frac{S_0}{K}\right)}{v\sqrt{T}}\right) \tag{3.4}$$

Definition 3.19. We call a universe *risk neutral* if at the expected value of the asset A and time period t, C(A,0) at t = 0 is the expected value of the asset at time t deducted to its present value, again utilizing the risk free rate r.

$$C(A,0) = e^{-rt}\mathbf{E}[C(A,t)]$$

$$\begin{aligned}C(S,0) &= e^{-rT}\mathbf{E}[C(S,T)] \\ &= e^{-rT}\mathbf{E}[max(S_T - K, 0)] \\ &= e^{-rT}\int_K^{\infty} \frac{1}{\sqrt{2\pi T}vx}(x-K)e^{\frac{-(\ln\frac{x}{S_0-\mu^2})^2}{2v^2T}} dx \\ &= e^{-rT}\int_K^{\infty} \frac{1}{\sqrt{2\pi T}v}e^{\frac{-(\ln\frac{x}{S_0-\mu^2})^2}{2v^2T}} dx - e^{-rT}\int_K^{\infty} \frac{1}{\sqrt{2\pi T}vx}Ke^{\frac{-(\ln\frac{x}{S_0-\mu^2})^2}{2v^2T}} dx\end{aligned}$$

We observe that the first integral is identical to the one we have in Lemma 3.17. Therefore, the first term simplifies to:

$$e^{-rT}S_0 e^{\mu+\frac{v^2T}{2}}\int_A^{\infty} \frac{1}{\sqrt{2\pi}}e^{\frac{-y^2}{2}} dy$$

Note that this is the same integral as in equation 3.15 except where the lower limit has been changed to A.

$$A = \frac{\ln \frac{K}{S_0} - \mu - v^2 T}{v\sqrt{T}}$$

We proceed to use equation 3.3 to find the value of μ , and then realize that the resulting integral represents the cumulative distribution function for the standard normal variable, so we have that:

$$\begin{aligned} S_0\bigg(1 - N\Big(\frac{\ln \frac{K}{S_0} - rT - \frac{v^2 T}{2}}{v\sqrt{T}}\Big)\bigg) &= S_0 N\bigg(- \frac{\ln \frac{K}{S_0} - rT - \frac{v^2 T}{2}}{v\sqrt{T}}\bigg) \\ &= S_0 N\bigg(\frac{\ln \frac{K}{S_0} + rT + \frac{v^2 T}{2}}{v\sqrt{T}}\bigg) \end{aligned}$$

and we have the first term of equation 3.4.

For our second term, we again look to simplify. To do this, we first start by letting

$$z = \frac{\ln \frac{x}{S_0} - \mu}{v\sqrt{T}}$$

then we can find the derivative of z to be

$$dz = \frac{dx}{xv\sqrt{T}}$$

Plugging this in to our second term, we have:

$$\begin{aligned} -e^{-rT} \int_K^\infty \frac{1}{\sqrt{2\pi T} v x} K e^{\frac{-(ln \frac{x}{S_0 - \mu^2})^2}{2v^2 T}} dx &= -e^{-rT} \int_{A+v\sqrt{T}}^\infty \frac{1}{\sqrt{2\pi}} (x-K) e^{\frac{-z^2}{2}} dz \\ &= -e^{-rT} K\bigg(1 - N\Big(A + v\sqrt{T}\Big)\bigg) \\ &= -Ke^{-rT} N\Big(- A - v\sqrt{T}\Big) \\ &= Ke^{\text{-rT}} N\bigg(\frac{rT + \frac{v^2 T}{2} + \ln(\frac{S_0}{K})}{v\sqrt{T}}\bigg) \end{aligned}$$

With that, we have found the second term of equation 3.4 and have completed our proof.

4. Derivation with Ito's Lemma

The derivation we just went over unquestionably leads to the correct result, however, we made many unjustified assumptions. First of all, we assumed that people are risk-neutral, which is plainly just not true. Most people will value an option for a given risky asset A and time period t at less than it's expected value $C(A, 0) < e^{-rt}\mathbf{E}[C(A, t)]$. It is perfectly fair to believe that they must be compensated for the risk they are taking when buying an option for such a volatile stock. This section will not only show the original method Black and Scholes employed to derive the Black-Scholes model, it will also justify why the assumption that people are risk-neutral is more than acceptable to make.

Definition 4.1. A stochastic process, W_t, for $t > 0$, is a *Brownian Motion* if $W_0 = 0$, and for any given t and s, in addition to the condition that $s < t$, W_t-W_s is continuous, and has a normal distribution with a variance value of $t - s$. In addition to this, the distribution of $W_t - W_s$ is independent of the behavior W_r for $r \leq s$

Definition 4.2. The family X of random variables X_t satisfies the stochastic differential equation (SDE):

$$dX_t = \mu(t, X_t)dt + \sigma(t, X_t)dW_t \tag{4.1}$$

if for any t,

$$X_{t+h} - X_t - h\mu(t, X_t) - \sigma(t, X_t)(W_{t+h} - W_t)$$

is a random variable where the mean and variance are $o(h)$ and W_t is a Brownian Motion

Definition 4.3. A stochastic process S_t is said to follow a *Geometric Brownian Motion* if it satisfies the stochastic differential equation

$$dS_t = \mu S_t dt + \sigma S_t dW_t \tag{4.2}$$

with μ and σ as constants and W_t a Brownian motion.

Finally, we can move on to the finally definition we need for this proof.

Definition 4.4. An *Ito Process*, X_t, is a process that satisfies the stochastic differential equation

$$dX_t = \mu(t)X_t dt + \sigma(t)X_t dW_t$$

Theorem 4.5. *(Ito's Lemma) Let X_t be an Ito process that satisfies equation 4.1, and let $f(x, t)$ be a twice-differentiable function, then $f(X_t, t)$ is an Ito process, and we have that*

$$d(f(X_t, t)) = \frac{\delta f}{\delta t}(X_t, t)dt + \frac{\delta f}{\delta X_t}dX_t + \frac{1}{2}\frac{\delta^2 f}{\delta {X_t}^2}d{X_t}^2 \tag{4.3}$$

where we define $d{X_t}^2$ by

$$dt^2 = 0 \tag{4.4}$$

$$dt dW_t = 0 \tag{4.5}$$

$$d{W_t}^2 = dt \tag{4.6}$$

Remark 4.6. Proving this theorem is beyond the scope of this paper, however a possible explanation for each equation shall be provided. For the first two (4.4 and 4.5), it is possible that dt is infinitesimal, or very small, therefore resulting in the zeroes. Now for equation 4.6, we can provide something nowhere near a proof, on the contrary actually something closer to an idea. We can imagine taking steps of length 1 or -1 (50/50 chance) at every second. Now, trivially we have the the expected value of the sum of steps is 0, as there is a 50/50 chance to get a 1 or -1. Now if we look at ${W_t}^2$, we have that it equals t due to the fact that there are t steps each of length one. With this, it seems plausible that $|W_t| = O(\sqrt{t})$, and so $W_t = O(\sqrt{\vec{\Delta} t})$. We can imagine $\vec{\Delta}$ heading towards 0, and that equation 4.6 resembles what we imagine as we think about $\vec{\Delta}$ as it goes to 0. The formal way to prove ito's lemma involves applying a second-order Taylor series expansion, see [1].

Finally, now that we are equipped with Ito's Lemma and the required knowledge, we will justify the use of the risk-free rate.

Theorem 4.7. *Given a European call option C(S,t), given an expiration of T, strike price of K, on a stock price S that follows a geometric Brownian Motion, and given that r represents the risk-free interest r, we have that*

$$\frac{\delta C}{\delta t}(S,t) + rS\frac{\delta C}{\delta S} + \frac{1}{2}\sigma^2 S^2 \frac{\delta^2 C}{\delta S^2}(S,t) = rC \tag{4.7}$$

Proof. Given by equation 4.3,

$$\begin{aligned} dC &= \frac{\delta C}{\delta t}(S,t)dt + \frac{\delta C}{\delta S}(S,t)dS + \frac{1}{2}\frac{\delta^2 C}{\delta S^2}(S,t)dS^2 \\ &= \left(\frac{\delta C}{\delta t}(S,t)dt + \frac{\delta C}{\delta S}(S,t)dS + \frac{1}{2}\sigma^2 S^2 \frac{\delta^2 C}{\delta S^2}(S,t)\right)dt + \sigma S \frac{\delta C}{\delta S}(S,t)dW_t \end{aligned}$$

since

$$dS = \mu S dt + \mu S dW_t$$

and

$$\begin{aligned} dS^2 &= \mu^2 S^2 dt^2 + \mu\sigma S^2 dt dW_t + \sigma^2 S^2 {dW_t}^2 \\ &= \sigma^2 S^2 dt \end{aligned}$$

This is given by equations 4.4, 4.5, and 4.6.

Now we proceed to considering a portfolio consisting of the call option in addition to $\propto$ stocks. The cost of the portfolio is $C + \propto S$. We use the same argument as before to get:

$$d(C + \propto S) = \left(\frac{\delta C}{\delta t}(S,t) + \frac{\delta C}{\delta S}(S,t) + \frac{1}{2}\sigma^2 S^2 \frac{\delta^2 C}{\delta S^2}(S,t)\right) + \propto \mu S\Bigg)dt + \mu S\left(\frac{\delta C}{\delta S}(S,t) + \propto\right)dW_t$$

Finally, we let $\propto = -\frac{\delta C}{\delta S}(S,t)$ to hedge away all risk in our portfolio, and we will observe in the following equation that the random component, dW_t, is now gone. This is very important, as now that it is risk free, it grows over time at risk-free rate r.

$$d(C + \propto S) = \left(\frac{\delta C}{\delta t}(S,t) + \frac{1}{2}\sigma^2 S^2 \frac{\delta^2 C}{\delta S^2}\right)dt$$

As mentioned before, as our portfolio is now risk-free, we have that

$$\frac{d}{dx}(C + \propto S) = r\left(C - S\frac{\delta C}{\delta S}\right)$$

and

$$r\left(C - S\frac{\delta C}{\delta S}\right) = \frac{\delta C}{\delta t}(S,t) + \frac{1}{2}\sigma^2 S^2 \frac{\delta^2 C}{\delta S^2}(S,t)$$

Rearranging gives us the Black-Scholes Model:

$$\frac{\delta C}{\delta t}(S,t) + rS\frac{\delta C}{\delta S} + \frac{1}{2}\sigma^2 S^2 \frac{\delta^2 C}{\delta S^2}(S,t) = rC$$

By hedging away all randomness, we proved that the portfolio had to risk, which shows why using risk-neutrality is perfectly fair. In addition to that, we finished our proof of the Black-Scholes Model and we will now proceed to talking about Put-Call Parity. ∎

5. Put-Call Parity

We will be dealing with the assumption of arbitrage-free pricing. While it exists in the real world, opportunities for them are rare and all they will do is give us a value for our option that won't be remotely close to what it would be when trading options in real life.

Let's consider a call and put option with the same strike price K and the same expiration date on some stock S, which pays no dividend. If we imagine assembling a portfolio with these two options, the payoff for this portfolio is S(T)-K, where the price of S is $S(t)$ at time t. Now let's imagine a second portfolio, with one share and borrowing K bonds, where we assume that the bond pays 1 dollar at it's maturity time of T. The payout of this second portfolio is also $S(T)-K$, as the bonds will each be worth 1 each to be worth K altogether, and the stock will of course be worth $S(T)$ at the expiration date.

This allows us to build the following relationship between the value of these instruments, known as the put-call parity:

$$C(t) - P(t) = S(T) = K * B(t,T)$$

From here, it is simple to calculate the price of a European put option with a given strike price K and expiration time T. From equation 2.4, we have that:

$$C = S_0 N\left(\frac{rT + \frac{v^2T}{2} + \ln\left(\frac{S_0}{K}\right)}{v\sqrt{T}}\right) - Ke^{-rT} N\left(\frac{rT + \frac{v^2T}{2} + \ln\left(\frac{S_0}{K}\right)}{v\sqrt{T}}\right)$$

We note that if the interest rate r is constant, which we have done constantly throughout this paper, we have that $B(0,T) = e^{-rT}$. With that, we proceed to solving for the value of the European put option:

$$\begin{aligned}
P(S,0) &= C(0) - S_0 + Ke^{-rT} \\
&= e^{-rT}\mathbf{E}[max(S_T - K, 0)] \\
&= S_0 N\left(\frac{rT + \frac{v^2T}{2} + \ln\left(\frac{S_0}{K}\right)}{v\sqrt{T}}\right) - S_0 + Ke^{-rT} + Ke^{-rT} N\left(\frac{rT + \frac{v^2T}{2} + \ln\left(\frac{S_0}{K}\right)}{v\sqrt{T}}\right) \\
&= -S_0\left(1 - N\left(\frac{rT + \frac{v^2T}{2} + \ln\left(\frac{S_0}{K}\right)}{v\sqrt{T}}\right)\right) + Ke^{-rT}\left(1 - N\left(\frac{rT + \frac{v^2T}{2} + \ln\left(\frac{S_0}{K}\right)}{v\sqrt{T}}\right)\right) \\
&= -S_0 N\left(-\frac{rT + \frac{v^2T}{2} + \ln\left(\frac{S_0}{K}\right)}{v\sqrt{T}}\right) + Ke^{-rT} N\left(-\frac{rT + \frac{v^2T}{2} + \ln\left(\frac{S_0}{K}\right)}{v\sqrt{T}}\right)
\end{aligned}$$

With that, we are done.

6. Extensions of the Black-Scholes Model

Using the original model, people expanded upon the original model created by Black and Scholes, extending the model into being able to calculate the values of more exotic options, such as options that last forever. We may also be changing the assumptions that we had set in the original proof, which will be noted before we begin the construction of the new model. In the following section, we will be exploring two different extensions of the model.

6.1. **Perpetuity.** The second of the extensions involve perpetuity, namely a "perpetual put". This is known as a "perpetual American option", also called an "XPO" or just a "perpetual option". These options are very rare and on the rare occasion that they are traded, are traded OTC, or over the counter. An over the counter transaction just means that it is a trade done directly between 2 parties, with no middle man or supervision of an exchange.

Definition 6.1. A *perpetual option* is plainly an option with no expiration date and no restrictions on when it can be exercised.

Theorem 6.2. *In a risk-neutral universe given that* $S \geq S_- = \frac{\lambda_2 K}{\lambda_2 - 1}$*, with an initial stock price S, the value P of an American perpetual put option with strike K, and no expiration time is:*

$$P(S) = \frac{K}{1-\lambda_2}\left(\frac{\lambda_2 - 1}{\lambda_2}\right)^{\lambda_2}\left(\frac{S}{K}\right)^{\lambda_2}$$

Proof. We start with the fact that the option is perpetual, and that it never expires. This means that the time decay of the option is zero, therefore leading to the Black-Scholes PDE (Partial Differential Equation) becoming an ODE (Ordinary Differential Equation):

$$\frac{1}{2}\sigma^2 S^2 \frac{d^2V}{dS^2} + (r-q)S\frac{dV}{dS} - rV = 0$$

We now define the lower exercise boundary as S_-, the lower boundary that is optimal for exercising the option, or in other words, the lower exercise boundary that will give us the best "value" for the option. The boundaries are listed below:

$$P(S_-) = K - S_-, V_S(S_- = -1, P(S) \leq K$$

Now, we solve the ODE and find that the solutions include a linear combination of any two solutions that are linearly independent:

$$P(S) = A_1 S^{\lambda_1} + A_2 S^{\lambda_2}$$

For values of S_- that are less than or equal to S, we can substitute this solution into the ODE for i = 1,2, giving us that:

$$\left[\frac{1}{2}\sigma^2\lambda_i(\lambda_i - 1) + (r-q)\lambda_i - r\right]S^{\lambda_i} = 0$$

We proceed by rearranging, giving that:

$$\frac{1}{2}\sigma^2\lambda_i^2 + \left(r - q - \frac{1}{2}\sigma^2\right)\lambda_i - r = 0$$

We now utilize the quadratic formula to solve for λ_i, and we find that the solutions are:

$$\lambda_1 = \frac{\left(r - q - \frac{1}{2}\lambda^2\right) + \sqrt{\left(r - q - \frac{1}{2}\lambda^2\right)^2 + 2\lambda^2 r}}{\lambda^2}$$

$$\lambda_2 = \frac{\left(r - q - \frac{1}{2}\lambda^2\right) - \sqrt{\left(r - q - \frac{1}{2}\lambda^2\right)^2 + 2\lambda^2 r}}{\lambda^2}$$

The only way for us to have a finite or valid solution for our perpetual put is to have $A_1 = 0$, as this is the only way we get the upper and lower finite bounds on the value of the put required for a solution. This leads to the solution $P(S) = A_2 S_2^{\lambda}$. Our first boundary condition, namely $P(S_-) = K - S_-$, allows us to know that:

$$P(S_-) = P(S) = A_2 S^{\lambda_2} = K - S_- \Longrightarrow A_2 = \frac{K - S_-}{(S_-)^{\lambda_2}}$$

Consequently, we find that the value of the perpetual put becomes:

$$P(S) = (K - S_-)\left(\frac{S}{S_-}\right)^{\lambda_2}$$

Our second boundary condition, namely $V_S(S_- = -1$, gives us the lower exercise boundary:

$$P_S(S_-) = \lambda_2 \frac{K - S_-}{S_-} = -1 \Longrightarrow S_- = \frac{\lambda_2 K}{\lambda_2 - 1}$$

Finally, we find that for $S \geq S_- = \frac{\lambda_2 K}{\lambda_2 - 1}$, the perpetual American put option, also known as an "XPO", is worth:

$$P(S) = \frac{K}{1 - \lambda_2}\left(\frac{\lambda_2 - 1}{\lambda_2}\right)^{\lambda_2}\left(\frac{S}{K}\right)^{\lambda_2}$$

And with that, we conclude our proof of the value of a American perpetual put.

■

6.2. **Dividends.** Finally, we will be investigating stocks that pay out dividends. We will be looking at 2 types of dividend payout methods and how they affect the value of a given option.

Remark 6.3. To simplify our model, for the following two subsections, we will define the Black-Scholes formula for call and put options as:

$$C(S_0, T) = e^{-rT}[FN(d_1) - KN(d_2)]$$
$$P(S_0, T) = e^{-rT}[KN(d_2) - FN(d_1)$$

where we have that F is is the forward price we defined as $S_0 e^{rt}$ and that

$$d_1 = \frac{1}{\sigma\sqrt{T}}\left[\ln\left(\frac{S_0}{K}\right) + \left(r + \frac{1}{2}\sigma^2\right)(T)\right]$$

and

$$d_2 = d_1 - \sigma\sqrt{T} = \frac{1}{\sigma\sqrt{T}}\left[\ln\left(\frac{S_0}{K}\right) + \left(r - \frac{1}{2}\sigma^2\right)(T)\right]$$

Definition 6.4. As a reminder, a *dividend* is a portion of profits distributed to shareholders by the corporation. This distribution typically happens quarterly.

6.2.1. *Continuous Yield Dividends.*

Definition 6.5. A stock paying *continuous yield dividends* is a stock that pays dividends out continuously. The amount of these dividends is proportional to the level of the stock price.

Remark 6.6. To make dealing with the continuous yielding dividend easier, we can define the current time as t and the expiration date of the option being T.

We begin by modelling the dividend payment paid over the time period $[t, t + dt]$ as:

$$qS_t dt$$

for a dividend yield with value q.

With this we have that the price of the option implied by the Black-Scholes model can be shown to be:

$$C(S_t, T) = e^{-r(T-t)}[FN(d_1) - KN(d_2)]$$
$$P(S_t, t) = e^{-r(T-t)}[KN(d_2) - FN(d_1)]$$

for call options and put options, respectively. In this formula, our new forward price can be represented by:

$$F = S_t e^{(r-q)(T-t)}$$

This can be seen in the terms d_1 and d_2, which are now:

$$d_1 = \frac{1}{\sigma\sqrt{T-t}}\left[\ln\left(\frac{S_t}{K}\right) + \left(r - q + \frac{1}{2}\sigma^2\right)(T-t)\right]$$

and

$$d_2 = d_1 - \sigma\sqrt{T-t} = \frac{1}{\sigma\sqrt{T-t}}\left[\ln\left(\frac{S_t}{K}\right) + \left(r - q - \frac{1}{2}\sigma^2\right)(T-t)\right]$$

With that, we have found the value of a stock that pays dividends out in a continuous manner.

6.2.2. *Discrete Proportional Dividends.*

Definition 6.7. A stock paying *discrete proportional dividends* is a stock that pays out dividends at pre-determined times $(t_1, t_2, t_3, ..., t_n)$ at regular intervals. Just like the continuous yield dividends, the amount of these dividends are proportional to the level of the stock price.

We can assume that a proportion δ of the given stock price is paid out at regular intervals $(t_1, t_2, t_3, ..., t_n)$. We can write the stock price as:

$$S_t = S_0(1-\delta)^{n(t)} e^{ut + \sigma W_t}$$

where n(t) represents the number of dividends that have been paid out by time t.
We can now recall the definition of call and put options that we talked about previously:

$$C(S_t, T) = e^{-r(T-t)}[FN(d_1) - KN(d_2)]$$
$$P(S_t, t) = e^{-r(T-t)}[KN(d_2) - FN(d_1)]$$

In this case, we have that the forward price F is:

$$F = S_0(1-\delta)^{n(T)}e^{rT}$$

With that, we have found the value of a stock that pays dividends out proportionally at regular intervals.

Acknowledgements. It is a pleasure to thank my mentor, Emma Cardwell, and my teacher, Simon Rubinstein-Salzedo, for all their amazing guidance through the process of writing this paper. I really enjoyed learning from them and greatly appreciate everything they did for me, answering all of my questions whether it had to do with math or with LaTex. Finally, I just want to thank Simon again for leading this amazing class and giving me an opportunity to work on this unique topic that I probably wouldn't have gotten to experience otherwise.

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