Model Theory and the Ax-Grothendieck Theorem

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Important Theorems in Model Theory

Completeness

Completeness Theorem (Gödel)

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- A theory is consistent iff it is satisfiable.

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- If $T \models \phi$, then $T \vdash \phi$.
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If T is satisfiable, then if $T \models \phi$, $T \not\models \neg\phi$. Therefore $T \nvdash \{\phi, \neg\phi\}$ for any sentence ϕ, and T is consistent.
Assume T is not satisfiable. Then $T \models (\phi \wedge \neg\phi)$. Thus, $T \vdash \{\phi, \neg\phi\}$, and T is inconsistent.

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If every finite subset of a theory T is satisfiable, then T is satisfiable.

Proof: Suppose T is not satisfiable. Then T is inconsistent, so we should be able to find $T \vdash \{\phi, \neg\phi\}$. Since this proof has to use finitely many formulas from T, there exists an inconsistent finite subset of T, by Completeness, also unsatisfiable.

Some Algebra

Groups, Rings and Fields

- A group G is a set in which you can perform one operation $*$, and for any $x, y \in G$, $x * y \in G$ (i.e. it is closed). Also, it has an identity element, and an inverse for every $x \in G$.

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- A ring is a group which is closed under two operations, and is abelian under one of them (i.e. $x * y = y * x$). It also has the associative and distributive properties.
- A field F is a ring which is abelian under two operations if the identity of one of them is removed from F. Fields can be infinite or finite. In this last case, the cardinality of it will be in the form p^n, meaning the n-th power of the prime p.

Algebraically Closed Fields

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Definition

The *characteristic* of an algebraically closed field is the least number p, such that $x + \overset{p \text{ times}}{\cdots} + x = 0$ for all x in such field (it is actually a bit more general than this, but this is the case we are interested in). If such p does not exist, we say the field has characteristic 0.

ACF_p

We will focus our attention in the theory of ACF. Notice that ACF is not complete, since there is no way of deducing its characteristic.

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Since neither ψ_p or $\neg\psi_p$ are implied by ACF, we add them as axioms. Let ACF_p be ACF plus ψ_p (ACF_0 will be ACF plus $\neg\psi_p$ for every prime p). We will be able to show ACF_p is complete, which will be very useful later on.

Completeness of ACF_p and a Useful Corollary

Vaught's Test

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We say a theory T is *κ-categorical* if any two models of T with cardinality κ are isomorphic.

Theorem (Vaught's Test)

Let T be a satisfiable $\mathcal{L}$-theory with no finite models that is κ-categorical for some infinite cardinal $\kappa \geq |\mathcal{L}|$. Then T is complete.

Completeness of ACF_p

Algebraically closed fields are determined up to isomorphism by their characteristic and transcendence degree (cardinality of the set of non-algebraic elements of the field). The cardinality of an algebraically closed field is $max\{\lambda, \aleph_0\}$, where λ is the transcendence degree.

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Hence, ACF_p is κ-categorical. By Vaught's Test, ACF_p is complete.

Lefschetz Principle

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Let ϕ be a sentence in the language of rings. The following statements are equivalent:

(i) ϕ is true in the complex numbers.

(ii) ϕ is true in some algebraically closed field of characteristic 0.

(iii) ϕ is true in every algebraically closed field of characteristic 0.

(iv) There are arbitrarily large primes p such that ϕ is true in some algebraically closed field of characteristic p.

(v) There exists m such that ϕ is true for all ACF_p, with $p > m$.

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Ax-Grothendieck Theorem

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Let $f : \mathbb{C}^n \longrightarrow \mathbb{C}^n$ be an injective polynomial. Then, f is surjective.

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2. Show that some algebraically closed fields imply ϕ.
3. Use the Lefschetz Principle and prove the theory of complex numbers implies ϕ.

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$$\forall a_{1,(0,0)}\forall a_{1,(0,1)}\forall a_{1,(0,2)}\forall a_{1,(1,0)}\forall a_{1,(1,1)}\forall a_{1,(2,0)}$$
$$\forall a_{2,(0,0)}\forall a_{2,(0,1)}\forall a_{2,(0,2)}\forall a_{2,(1,0)}\forall a_{2,(1,1)}\forall a_{2,(2,0)}$$
$$\Bigg[\forall x_1\forall x_2\forall y_1\forall y_2\Bigg(\Bigg(\sum a_{1,(i_1,i_2)}x_1^{i_1}x_2^{i_2} = \sum a_{1,(i_1,i_2)}y_1^{i_1}y_2^{i_2} \wedge \sum a_{2,(i_1,i_2)}x_1^{i_1}x_2^{i_2} =$$
$$\sum a_{2,(i_1,i_2)}y_1^{i_1}y_2^{i_2}\Bigg) \longrightarrow x_1 = y_1 \wedge x_2 = y_2\Bigg) \longrightarrow$$
$$\forall z_1\forall z_2\exists x_1\exists x_2\Bigg(\sum a_{1,(i_1,i_2)}x_1^{i_1}x_2^{i_2} = z_1 \wedge \sum a_{2,(i_1,i_2)}x_1^{i_1}x_2^{i_2} = z_2\Bigg)\Bigg].$$

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Notice the theorem is trivial if we change the domain and range of the function to some finite field.

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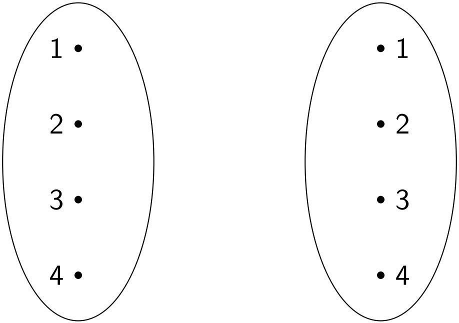

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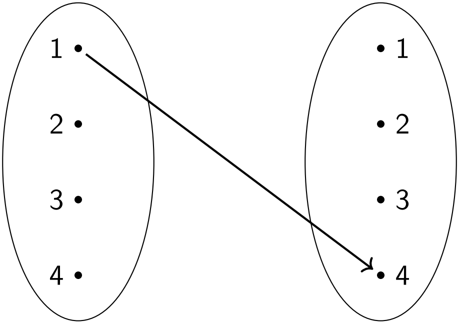

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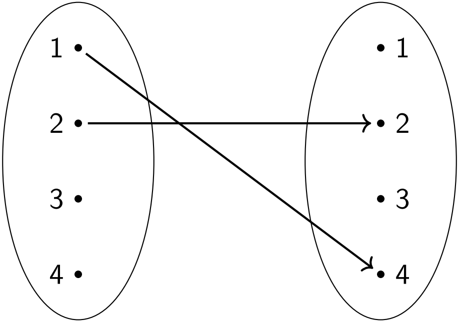

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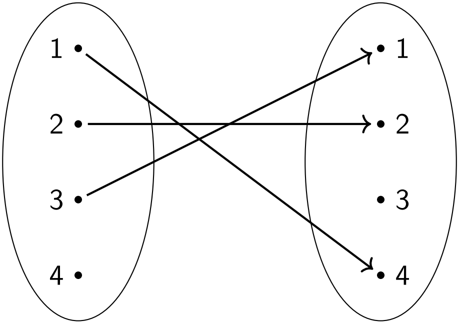

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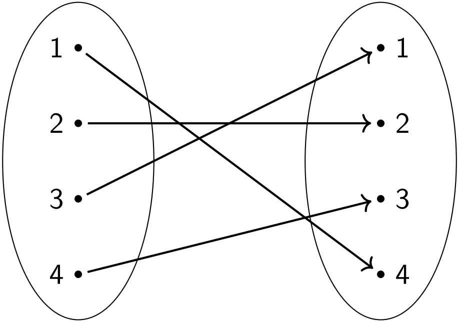

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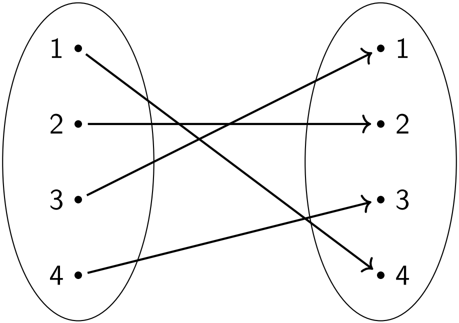

Since it holds in any finite field, then it holds in any increasing union of finite fields. Then we can prove it will hold in some algebraically closed fields. More specifically, $ACF_p \models \phi_{n,d}$ with arbitrarily large p.

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Thus, this is equivalent to saying $\mathbb{C} \models \phi_{n,d}$. Contradiction!

The Non-Standard Method

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Example

Non-standard analysis is a non-standard model.

Löwenheim–Skolem Theorems

For infinite models, Löwenheim–Skolem Theorems guarantee us the existence of non-standard models.

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Definition

If we have structures $\mathcal{M}$ and $\mathcal{N}$ with $M \subset N$, and for all formulas ϕ made up from elements of M, $\mathcal{M} \models \phi \iff \mathcal{N} \models \phi$, then $\mathcal{M}$ is the elementary substructure of $\mathcal{N}$ (and $\mathcal{N}$ is the elementary extension of $\mathcal{M}$).

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Löwenheim–Skolem Theorems

In a language $\mathcal{L}$, for every infinite $\mathcal{L}$-structure $\mathcal{M}$ and every infinite cardinal number $\kappa \geq |\mathcal{L}|$, there is an $\mathcal{L}$-structure $\mathcal{N}$ such that $|N| = \kappa$ and such that:

- if $\kappa < |M|$ then $\mathcal{N}$ is an elementary substructure of $\mathcal{M}$;
- if $\kappa > |M|$ then $\mathcal{N}$ is an elementary extension of $\mathcal{M}$.

The End

That is all, thank you!