MODEL THEORY AND THE AX-GROTHENDIECK THEOREM

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Abstract. In this paper we will explain basic concepts of first order logic and model theory. We will introduce some notions from abstract algebra that will allow us to give a proof for the Ax-Grothendieck theorem, and give the reader a perspective on the non-standard method. This paper relies heavily on [Mar00], but our goal was to show more detailed and clearer proofs, so as to maximize the reader's understanding.

1. Introduction

Model theory is a branch of mathematical logic, which regards mathematical structures (such as groups, fields or graphs) and relates them to formal theories, using first order logic (and first order languages). This field relies in the concept of truth, and most of the work we will do is proving some theories are true within a structure (i.e. the structure is a model of the theory), or starting with a structure and figuring out what could be true in it. We could also define model theory as the study of formal languages and their interpretations, or at least it started like that. Nowadays, "model theory is the study of the interpretation of any language, formal or natural, by means of set-theoretic structures, with Alfred Tarski's truth definition as a paradigm"(Wilfrid Hodges).

As in every branch of mathematical logic, model theory means abstraction. As we discuss notions as consistency, satisfiability, decidability and completeness, we will see that the choice of the language we choose to express mathematical ideas may change mathematics. How and where we interpret a language affects the truth of a certain statement. Despite this, model theory finds a clear pathway towards classical mathematics, as opposed to the rest of mathematical logic branches. Logic results apply only in logic, and classical math results apply only in classical math. Model theory is a sort of superposition, an intermediary that can translate results from one field to the other. This is what we will be exploring in this paper. We will begin by proving two theorems that work as cornerstones of model theory (the Completeness Theorem, which states a theory is consistent if an only if it has a model, and the Compactness Theorem, which states that if every finite subset of a theory has a model, then the theory istelf has a model). Then, using these theorems, we will see the connections abstract algebra shares with model theory, and reach our main theorem: the Ax-Grothendieck Theorem. This theorem states that if an n-dimensional complex polynomial function is injective, then it is surjective. This result alone does not tell us much, although it might be weird for the reader to see it, as it seems completely unrelated to the subject. The thing that is actually interesting is the proof that it uses, which illustrates perfectly this overlapping between model theory and the rest of mathematics (also, the author would like to add that it is beautiful). Lastly, we introduce the Löwenheim Skolem Theorems, which proves the existence of non-standard models for infinite models, and introduce exciting possibilities.

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We could choose a lot of starting dates for model theory, so we will just stick to one. We will consider model theory started out in 1915, when Leopold Löwenheim published a less general case of the Downward Löwenheim-Skolem Theorem, making this the first significant result in the field of study. Afterwards, the famous mathematician Kurt Gödel made major advancements in logic. In 1930, he formulated the Completeness Theorem, and the Compactness Theorem as a lemma of it. Anatoly Maltsev was able to generalize all these theorems between the years 1936-1941. The name "model theory "was coined by Alfred Tarski in 1954. He is a very important figure, since he developed model theory as an independent discipline. He set the foundations for the subject with his works on logical consequence, deductive systems, the algebra of logic, the theory of definability, and the semantic definition of truth. He achieved admirable results, such as proving the decidability of the real closed fields, and as a corollary, the decidability of euclidean geometry. Here, model theory took a turn. It started drifting away from logic, and venturing onto other branches of mathematics. Ultraproducts became a popular tool, and Abraham Robinson was able to develop non-standard analysis in the 1960s (this work was followed by Howard Jerome Keisler). In this decade, significant connections between model theory and abstract algebra (more specifically, algebraic classes) were made by James Ax. Now, we reach the second turning point in model theory. Saharon Shelah was able to develop stability theory, which gave place to a whole new class of concepts and questions, which are the topics for which model theory is relevant today.

Now, regarding the paper. In Section 2 we will introduce some basic concepts about first order languages, as well as mathematical structures. In Section 3 we define the concept of truth, and we start applying it to models of theories. In Section 4 we will prove the two main theorems of model theory we talked about: Completeness and Compactness. In Section 5 we prove a generalization of the Compactness Theorem. Using this result, we can move on to Section 6, and transition towards abstract algebra and algebraically closed fields. Then, we prove our main theorem in Section 7: the Ax-Grothendieck Theorem. In Section 8 we give proofs for the Löwenheim-Skolem Theorems. Section 9 is an appendix which covers the following topics: induction on formulas, cardinals, abstract algebra and Zorn's Lemma, in a very high pace. Whenever you feel lost, we recommend you to check the appendix.

2. Preliminaries

2.1. **First Order Languages.**

Notation. A *first order language* $\mathcal{L}$ consists of a set of symbols and the formulas we con build with them (following certain rules). These symbols can be categorized in the following way:

(1) Logical symbols
- Parentheses: $(,)$. They eliminate ambiguity, allowing only one possible interpretation of a formula. One should read formulas from left to right, and assume connective symbols apply only to the symbol (or formula between parentheses) immediately to the right.
- Connective symbols: $\wedge, \vee, \longleftrightarrow, \longrightarrow, \neg$. These are, respectively, and, or, iff, implies, not.
- Variables: $v_1, v_2 \ldots, v_n$, one for each positive integer n. We can have infinitely many variables. We will also use for variables the letters $x, y, z, \ldots$.

- Equality symbol (not necessarily): $=$. It is technically a predicate symbol, but it falls into this category since we will consider the axioms of equality as rules of logic. It is possible to consider it a non logical symbol, but it will affect its behavior under translations into informal language.

(2) Parameters
- Quantifier symbols: $\exists, \forall$. These mean, respectively, there exists, for all.
- Predicate symbols: They imply a relation between some elements. An n-ary predicate symbol R describes a relation between n elements, we will refer to n as the *arity* of R.
- Constant symbols: Some set (possibly empty) of symbols.
- Function symbols: They work the same way as all functions. An n-ary function f is applied to n elements, as with predicates.

Example. From the English language, we can translate the phrase "every person who is blond-haired is not red-haired" to a first order language where the variables correspond to people, B is an unary predicate indicating a person has blond hair and R is an unary predicate indicating a person has red hair. It would look like this

$$\forall x(Bx \longrightarrow (\neg Rx)).$$

Example. In the language of elementary number theory, where variables correspond to natural numbers, the number 0 is the constant symbol, the functions are $+$, $\cdot$, S, exp (addition, multiplication, successor and exponentiation, respectively), and we have equality and the predicate symbol $<$, meaning less-than. We can write the following statements:

- $\exists x(y = x + x)$. This tells us that y is even.
- $x = S0$. This means $x = 1$
- $\forall x(x = y \cdot z \longrightarrow y = S0 \vee z = S0)$. This translates to "all natural numbers are either prime or 1" (clearly false).

Definition 2.1. A finite sequence of symbols will be called an *expression*.

Definition 2.2. We define a *term* to be an expression we can get from the constant symbols and variables, by applying zero or more times a function symbol.

Definition 2.3. An *atomic formula* is an expression which follows the form

$$R_{t_1 \cdots t_n}.$$

Where R is an n-ary predicate symbol and $t_1, \ldots, t_n$ are terms.

Definition 2.4. A *well formed formula* is an expression built by atomic formulas and the use (zero or more times) of connective symbols. From this point forward we will refer to well formed formulas as formulas.

In formulas, variables may or may not appear tied to a quantifier. If in the formula ϕ a variable v_1 is not tied to any quantifier we say v_1 *occurs free* in ϕ (otherwise we say it is *bound*). If variables $v_1, \ldots, v_n$ occur free in ϕ, we can indicate that by writing $\phi(v_1, \ldots, v_n)$.

Definition 2.5. A formula with no free variables is called a *sentence*.

It is possible for us to turn a formula ϕ with free variables into a sentence using a function $s : V \longrightarrow S$, where s will translate the set V of all variables into some set S, and each variable x will be translated to $s(x)$. When this happens, we indicate it as $\phi_{[s]}$.

Example. Again, with the language of elementary number theory, we can get every natural number using the constant 0 and successor function. These are all terms. Moreover, the expression $Sx < x + y$ is an atomic formula. We can keep building and get a more complex formula, like $Sx < x+y \longleftrightarrow S0 < y$. If we quantify all variables we end up with a sentence:

$$\forall x \forall y (Sx < x + y \longleftrightarrow S0 < y).$$

2.2. **Structures.** *Mathematical structures* are described by languages. We could say a structure is an interpretation of a language. Essentially, a structure is a set with functions and relations, which interprets a language $\mathcal{L}$. We call this an $\mathcal{L}$-*structure*.

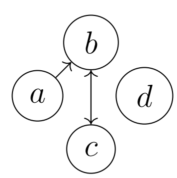
Definition 2.6. Within a language $\mathcal{L}$ where we have a set $\mathcal{F}$ of function symbols f, a set $\mathcal{R}$ of predicate symbols R and a set $\mathcal{C}$ of constant symbols, an $\mathcal{L}$-structure $\mathcal{M}$ is given by:

- A set of elements to which the universal quantifier $\forall$ refers to, called the *underlying set* of $\mathcal{M}$. we will refer to it as M. The amount of elements in the underlying set is called the *cardinality* of the set, and it will be denoted by $|M|$.
- A function $f^{\mathcal{M}} : M^n \longrightarrow M$ for each $f \in \mathcal{M}$, where f is n-ary.
- A relation $R^{\mathcal{M}}$ for each $R \in \mathcal{R}^n$, where R is n-ary.
- An element $c^{\mathcal{M}}$ for each $c \in \mathcal{C}$.

We refer to $f^{\mathcal{M}}$, $R^{\mathcal{M}}$, and $c^{\mathcal{M}}$ as the *interpretations* of symbols f, R and c.

Notation. The structure $\mathcal{M}$ can be denoted by writing $(M; R^{\mathcal{M}}, f^{\mathcal{M}}, c^{\mathcal{M}})$.

Example. We can take the finite structure $\mathcal{G}$, $(\{a, b, c, d\}; E^{\mathcal{G}})$, where E is a binary predicate symbol, and $E^{\mathcal{G}} = (\{a, b\}, \{b, c\}, \{c, b\})$. It is possible to interpret it as a directed graph, where the elements of the underlying set are vertices and the relation $E^{\mathcal{G}}$ indicates an edge between two vertices.



Definition 2.7. Let $\mathcal{M}$ and $\mathcal{N}$ be $\mathcal{L}$-structures and $\sigma : M \longrightarrow N$ an injective function. If σ satisfies

(1) $\sigma(f^{\mathcal{M}}(a_1, \ldots, a_n)) = f^{\mathcal{N}}(\sigma(a_1), \ldots, \sigma(a_n))$ for all n-ary $f \in \mathcal{F}$ and $a_1, \ldots, a_n \in M$
(2) $(a_1, \ldots, a_n) \in R^{\mathcal{M}}$ iff $(\sigma(a_1), \ldots, \sigma(a_n)) \in R^{\mathcal{N}}$ for every n-ary $R \in \mathcal{R}$ and $a_1, \ldots, a_n \in M$
(3) $\sigma(c^{\mathcal{M}}) = c^{\mathcal{N}}$ for all $c \in \mathcal{C}$

then we say σ is an $\mathcal{L}$-*embedding*. A bijective $\mathcal{L}$-embedding is an $\mathcal{L}$-*isomorphism*. Whenever there exists an embedding from M to N and $M \subseteq N$, we call $\mathcal{M}$ a *substructure* of $\mathcal{N}$, or $\mathcal{N}$ an *extension* of $\mathcal{M}$.

Example. $(\mathbb{Z}; +, 0)$ is a substructure of $(\mathbb{R}; +, 0)$. Let us define $\sigma : \mathbb{Z} \longrightarrow \mathbb{R}$, and let $\sigma(x) = x$. Then, we can see that σ is an embedding, because it is injective and it satisfies all the conditions stated before:

(1) $\sigma(a_1 + a_2) = \sigma(a_1) + \sigma(a_2)$ for every $a_i \in \mathbb{Z}$.
(2) There are no relations in these structures, we could count equality as one but we will not do that because of what we stated in the beginning of the section.
(3) $\sigma(0) = 0$.

Since σ is an embedding and $\mathbb{Z} \subseteq \mathbb{R}$, then our statement is true.

3. Models and Theories

In any $\mathcal{L}$-structure, sentences can be true or false. Whenever a sentence ϕ is true in a structure $\mathcal{M}$ we will say that $\mathcal{M}$ *satisfies* ϕ, and write this as $\mathcal{M} \models \phi$. We will now define more generally and rigorously the concept of *truth*. Let $\phi(v_1, \ldots, v_n)$ be a formula, and $(v_1, \ldots, v_n) = \bar{v}$ its free variables. Let $\bar{a} = (a_1, \ldots, a_n) \in M^n$.

Definition 3.1. We define $\mathcal{M} \models \phi(\bar{a})$ as it follows:

(1) Atomic formulas
  (i) Let t_i be a term, if ϕ is $t_1 = t_2$, then $\mathcal{M} \models \phi(\bar{a})$ iff $t_1^{\mathcal{M}}(\bar{a}) = t_2^{\mathcal{M}}(\bar{a})$.
  (ii) If ϕ is $R(t_1, \ldots, t_n)$, then $\mathcal{M} \models \phi(\bar{a})$ iff $(t_1^{\mathcal{M}}(\bar{a}), \ldots, t_n^{\mathcal{M}}(\bar{a})) \in R^{\mathcal{M}}$.
(2) Other formulas
  (i) If ϕ is $\neg\psi$, then $\mathcal{M} \models \phi(\bar{a})$ iff $\mathcal{M} \not\models \psi(\bar{a})$.
  (ii) If ϕ is $\alpha \longrightarrow \beta$, then $\mathcal{M} \models \phi(\bar{a})$ iff $\mathcal{M} \not\models \alpha(\bar{a})$ or $\mathcal{M} \models \beta(\bar{a})$ (or both).
  (iii) If ϕ is $\forall v_i\ \psi(\bar{v}, v_i)$, then $\mathcal{M} \models \phi(\bar{a})$ iff $\mathcal{M} \models \psi(\bar{a}, b)$, for all $b \in M$.

Remark 3.2. This is a definition by induction. If you are not familiar with them, we strongly recommend you to read the appendix about induction on formulas. Also, read carefully the proof for the next proposition.

Definition 3.3. If $\mathcal{M}$ and $\mathcal{N}$ are both structures, and for all $\mathcal{L}$-sentences ϕ

$$\mathcal{M} \models \phi \iff \mathcal{N} \models \phi$$

then we say $\mathcal{M}$ and $\mathcal{N}$ are *elementarily equivalent*, and write $\mathcal{M} \equiv \mathcal{N}$.

Example. In the language of number theory, the structure of natural numbers is elementary equivalent to the structure of even natural numbers. It is clear that any sentence true in one is true in the other. This can be explained by the following result.

Proposition 3.4. *If $\mathcal{M}$ and $\mathcal{N}$ are isomorphic, then they are elementarily equivalent.*

Proof. We will show this by induction on formulas. Let $\sigma : M \longrightarrow N$ be an isomorphism. First we will prove that $\sigma(t^{\mathcal{M}}(\bar{a})) = t^{\mathcal{N}}(\sigma(\bar{a}))$, which will be useful to prove our statement holds in all atomic formulas. Then we will show that if our statement holds in all atomic formulas ϕ and ψ, then it also holds in the formulas $\neg\phi$, $\phi \longrightarrow \psi$ and $\forall x\phi$. Thus, it will hold in all formulas.

Claim 3.5. *Let t be a term with free variables $(v_1, \ldots, v_n) = \bar{v}$ and $\bar{a} = (a_1, \ldots, a_n) \in M$. Let $\sigma(\bar{a})$ denote $(\sigma(a_1), \ldots, \sigma(a_n))$. Then, $\sigma(t^{\mathcal{M}}(\bar{a})) = t^{\mathcal{N}}(\sigma(\bar{a}))$.*

Proof. We show this by induction on terms.

(i) If $t = c$, then $\sigma(t^{\mathcal{M}}(\bar{a})) = \sigma(c^{\mathcal{M}}) = c^{\mathcal{N}} = t^{\mathcal{N}}(\sigma(\bar{a}))$. This follows from item (3) in 2.7.
(ii) If $t = v_i$, then $\sigma(t^{\mathcal{M}}(\bar{a})) = \sigma(a_i) = t^{\mathcal{N}}(\sigma(a_i))$.
(iii) If $t = f(t_1 \ldots t_n)$, then

$$\begin{aligned} \sigma(t^{\mathcal{M}}(\bar{a})) &= \sigma(f^{\mathcal{M}}(t_1^{\mathcal{M}}(\bar{a}), \ldots, t_n^{\mathcal{M}}(\bar{a})) \\ &= f^{\mathcal{N}}(\sigma(t_1^{\mathcal{N}}(\bar{a})), \ldots, \sigma(t_n^{\mathcal{N}}(\bar{a}))) \\ &= t^{\mathcal{N}}(\sigma(\bar{a})). \end{aligned}$$

This follows from item (1) in 2.7.

■

Now, we can show the statement holds in any atomic formula ϕ:

(i) If ϕ is $t_1 = t_2$, then

$$\begin{aligned}\mathcal{M} \models \phi(\bar{a}) &\iff t_1^{\mathcal{M}}(\bar{a}) = t_2^{\mathcal{M}}(\bar{a}) \\ &\iff \sigma(t_1^{\mathcal{M}}(\bar{a})) = \sigma(t_2^{\mathcal{M}}(\bar{a})) \\ &\iff t_1^{\mathcal{N}}(\sigma(\bar{a})) = t_2^{\mathcal{N}}(\sigma(\bar{a})) \\ &\iff \mathcal{N} \models \phi(\sigma(\bar{a})).\end{aligned}$$

(ii) If ϕ is $R(t_1, \ldots, t_n)$, then

$$\begin{aligned}\mathcal{M} \models \phi &\iff (t_1^{\mathcal{M}}(\bar{a}), \ldots, t_n^{\mathcal{M}}(\bar{a})) \in R^{\mathcal{M}} \\ &\iff (\sigma(t_1^{\mathcal{N}}(\bar{a})), \ldots, \sigma(t_n^{\mathcal{N}}(\bar{a}))) \in R^{\mathcal{N}} \\ &\iff R^{\mathcal{N}} \models \phi(\sigma(\bar{a})).\end{aligned}$$

Thus, the statement is true for all atomic formulas. Our inductive hypothesis will be that this property also holds in ψ and β. We will now prove that if our hypothesis is true, then the statement holds as we build bigger and bigger formulas:

(i) If ϕ is $\neg\psi$, then $\mathcal{M} \models \phi(\bar{a}) \iff \mathcal{M} \not\models \psi(\bar{a}) \iff \mathcal{N} \not\models \psi(\sigma(\bar{a})) \iff \mathcal{N} \models \phi(\sigma(\bar{a}))$.

(ii) If ϕ is $\psi \longrightarrow \beta$, then $\mathcal{M} \models \phi(\bar{a})$ iff either $\mathcal{M} \not\models \psi(\bar{a})$ or $\mathcal{M} \models \beta(\bar{a})$ (or both). This means either

$$\begin{aligned}\mathcal{M} \models \phi(\bar{a}) &\iff \mathcal{M} \not\models \psi(\bar{a}) \\ &\iff \mathcal{N} \not\models \psi(\sigma(\bar{a})) \\ &\iff \mathcal{N} \models \phi(\sigma(\bar{a})),\end{aligned}$$

or

$$\begin{aligned}\mathcal{M} \models \phi(\bar{a}) &\iff \mathcal{M} \models \beta(\bar{a}) \\ &\iff \mathcal{N} \models \beta(\sigma(\bar{a})) \\ &\iff \mathcal{N} \models \phi(\sigma(\bar{a})).\end{aligned}$$

As we can see, the reasoning is similar to the first item.

(iii) If ϕ is $\forall x\psi$, then $\mathcal{M} \models \phi(\bar{a}) \iff \mathcal{M} \models \psi(\bar{a}, b)$ for all $b \in M \iff \mathcal{N} \models \psi(\sigma(\bar{a}), c)$ for all $c \in N \iff \mathcal{N} \models \phi(\bar{a})$.

Since we have proven our statement holds in any atomic formula, and we have proven it also holds in any formula built by formulas in which it holds, then it holds for all $\mathcal{L}$-formulas. ■

Definition 3.6. An $\mathcal{L}$-*theory* T is a set of $\mathcal{L}$-sentences in a language $\mathcal{L}$. We say $\mathcal{M}$ is a *model* of T and write $\mathcal{M} \models T$ iff for every sentence $\phi \in T$, $\mathcal{M} \models \phi$.

Definition 3.7. If a theory has a model, we say it is *satisfiable*.

Example. Let us take the formula $x^2 - y = 1$. This is satisfiable, since it has a model. If we say $x = 1$ and $y = 0$, then it is true. On the contrary $\forall x(x \neq x)$ is unsatisfiable, since it cannot be true in any structure.

Definition 3.8. Let T be a theory and ϕ be a formula. We say T *logically implies* ϕ if $\mathcal{M} \models \phi$ whenever $\mathcal{M} \models T$.

Example. Let us take a theory made up of one formula: $2x = 2$. Then, it logically implies $x = 1$, since every model of $2x = 2$ is a model of $x = 1$.

Whenever we want to show that $T \models \phi$, we are faced to checking if every model of T satisfies ϕ. Since doing this could take a lifetime, what we do is give an informal mathematical proof. A proof of ϕ from T consists in a finite chain of formulas $\alpha_1, \ldots, \alpha_n$, where $\alpha_1 \in T \cup \Lambda$, α_n is ϕ, and for each α_k, either $\alpha_k \in T \cup \Lambda$ or α_k is obtained because α_i and $\alpha_i \longrightarrow \alpha_k$ are formulas previous to α_k in said chain. Λ denotes the set of *axioms* (which could be infinite). We call this way of inferring new formulas *modus ponens*.

Definition 3.9. A sentence ϕ which can be obtained in this manner from $T \cup \Lambda$ is a *theorem* of T, written $T \vdash \phi$. The formulas $\alpha_1, \ldots, \alpha_n$ are the *deduction* of ϕ from T.

4. Basic Results in Model Theory

Lemma 4.1. *If ϕ is a logical axiom, then ϕ is valid, meaning $\varnothing \models \phi$. This is equivalent to saying any structure and any variable assignation satisfies ϕ.*

We will not be giving a proof in this paper. This lemma follows from a categorization of the kinds of logical axioms, and afterwards proving the lemma for each category. A proof can be found in [End01], section 2.5 page 131.

Definition 4.2. A theory T is said to be *consistent* if and only if there are not any formulas ϕ such that $T \vdash \phi$ and $T \vdash \neg\phi$. If such formula exists, then T is *inconsistent*.

Theorem 4.3 (Soundness Theorem)**.** *If $T \vdash \phi$, then $T \models \phi$*

Proof. We show this by induction.

(i) If ϕ is an axiom, then by 4.1, $T \models \phi$.
(ii) If $\phi \in T$, $T \models \phi$, since it is clear that every model that satisfies T, by force has to satisfy ϕ.
(iii) If ϕ is obtained from ψ and $\psi \longrightarrow \phi$ by modus ponens, then we know by our inductive hypothesis that $T \models \psi$ and $T \models \psi \longrightarrow \phi$. Therefore, $T \models \phi$, since if this does not happen, then $T \not\models \psi \longrightarrow \phi$.

■

Theorem 4.4 (Gödel's Completeness Theorem)**.** *An $\mathcal{L}$-theory T is consistent if and only if it is satisfiable.*

This proof contains assumptions that will be justified later on. If this is your first reading, we recommend you to move forward and read the full proof later.

Proof. Suppose T is inconsistent. Then, $T \vdash \phi$ and $T \vdash \neg\phi$. By the Soundess Theorem, $T \models \{\phi, \neg\phi\}$. Therefore, T cannot have a model, and it is unsatisfiable. Now, assume T is consistent. We extend $\mathcal{L}$ and T to $\mathcal{L}^*$ and T^* just like in 5.8. By 5.3, T^* has a model $\mathcal{M}^*$. Reduce $\mathcal{M}^*$ to $\mathcal{M}$, where we limit the model to the language $\mathcal{L}$. Since formulas in T only involve constants in $\mathcal{L}$, then $\mathcal{M} \models T$. ■

Lemma 4.5. *If $T \cup \alpha \vdash \beta$, then $T \vdash \alpha \longrightarrow \beta$.*

Proof. If the deduction of β does not use α, then β can be deducted from just formulas in T, so $T \vdash \beta$, and therefore $T \vdash \alpha \longrightarrow \beta$. If the deduction of β uses α, we have two cases. Either $T \vdash \alpha$, and then $T \vdash \beta$ since we can obtain α from T, and therefore we can deduct β, or $T \nvdash \alpha$. Thus, $T \vdash \alpha \longrightarrow \beta$. ■

Corollary 4.6. *Let T be a theory and ϕ an $\mathcal{L}$-sentence. Then, $T \models \phi$ if and only if $T \vdash \phi$.*

Proof. Because of Theorem 4.3, we only have to prove the converse: if $T \models \phi$ then $T \vdash \phi$. Assume $T \models \phi$. $\mathcal{M} \models T$ implies $\mathcal{M} \models \phi$. Therefore, $\mathcal{M} \not\models \neg\phi$, which means $T; \neg\phi$ is unsatisfiable ($T; \neg\phi$ means the theory T plus the formula $\neg\phi$), and hence, inconsistent. If $T; \neg\phi$ is inconsistent, then $T; \neg\phi \vdash \{\alpha, \neg\alpha\}$. By 4.5, $T \vdash (\neg\phi \longrightarrow \alpha)$ and $T \vdash (\neg\phi \longrightarrow \neg\alpha)$ for some formula α. $\{\neg\phi \longrightarrow \alpha, \neg\phi \longrightarrow \neg\alpha\}$ implies ϕ, thus $T \vdash \phi$. ■

Theorem 4.7 (Compactness Theorem)**.** *If every finite subset of a theory T is satisfiable, then T is satisfiable.*

Proof. If T is finite, then it is trivial to see that T is satisfiable, given that T is a subset of itself. Suppose T is infinite. If T is unsatisfiable, then by Theorem 4.3 it is inconsistent. Let us say $T \vdash \alpha$ and $T \vdash \neg\alpha$. Let $R \subset T$ be the set of formulas from which one can deduct α and $S \subset T$ be the of formulas from which one can deduct $\neg\alpha$. Both R and S are finite, so $R \cup S$ is also finite, and it is a subset of T. Because of this, $R \cup S$ is satisfiable, and thus, consistent. But we have stated that $R \cup S \vdash \{\alpha, \neg\alpha\}$. Contradiction! ■

This Theorem may seem simple, but it will be one of the most important tools in this paper. Whenever every finite subset of a theory T is satisfiable, we will say T is *finitely satisfiable.*

Proposition 4.8. *For T a theory, if $T \models \phi$, then there exists a finite $\Delta \subseteq T$ such that $\Delta \models \phi$.*

Proof. If $\Delta \not\models \phi$, then $\Delta; \neg\phi$ is satisfiable. Consider T a union of satisfiable subsets, since $\Delta; \neg\phi$ is satisfiable, $T \cup \neg\phi$ is satisfiable, because of the Compactness Theorem. Thus $T \not\models \phi$. Contradiction! ■

5. The Compactness Theorem Revisited

We will proceed to prove a stronger version of the Compactness Theorem, with a proof based on one of the Completeness Theorem given by Leon Henkin in [Hen49]. We will show that for every finitely satisfiable theory it is possible to construct a model by adding constant symbols to the language, so that every element of the model is named by a constant symbol. It will take long to prove this, if the proof seems overwhelming we recommend to skip forward, and return to it later.

Definition 5.1. We say that an $\mathcal{L}$-theory T has the *witness property* if whenever $\phi(v)$ is a formula with one free variable v, then there is a constant symbol $c \in \mathcal{L}$ such that $(\exists v \phi(v)) \longrightarrow \phi(c)) \in T$.

An $\mathcal{L}$-theory T is *maximal* if for all ϕ either $\phi \in T$ or $\neg\phi \in T$.

Lemma 5.2. *Let T be a maximal and finitely satisfiable $\mathcal{L}$-theory. If $\Delta \subseteq T$ is finite and $\Delta \models \psi$, then $\psi \in T$.*

Proof. Suppose not. Then, because T is maximal, $\neg\psi \in T$, and thus $\Delta \cup \neg\psi$ is satisfiable, and this means $\Delta \not\models \psi$. Contradiction! ■

Proposition 5.3. *Let T be a maximal and finitely satisfiable $\mathcal{L}$-theory with the witness property. Let the language $\mathcal{L}$ have at most κ constant symbols, then T has a model $\mathcal{M}$ which cardinality is less than or equal to κ, $|M| \leq \kappa$.*

Proof. Let $\mathcal{C}$ be the set of constant symbols in $\mathcal{L}$. For $c, d \in \mathcal{C}$, we say $c \sim d$ if $c = d \in T$.

Claim 5.4. *$c \sim d$ is an equivalence relation.*

Proof. $c = c$ is in T. Let us say $c = d$ and $d = e$ are in T, we can see that since the sentences $c = c$, $d = d$, $c = d$ are in T, and those sentences imply $d = c$, then by 5.2 $d = c \in T$. With the same argument, $c = e \in T$. ■

The underlying set of our model will be $M = \mathcal{C}/\sim$, the equivalence classes of $\mathcal{C}$ mod $\sim$. Since there cannot be more equivalence classes than constant symbols, then $|M| \leq \kappa$. Let c^* denote the equivalence class of c, our structure will interpret c as is equivalence class $c^* = c^{\mathcal{M}}$. Now we will see how we will interpret predicate and function symbols. Suppose R is an n-ary predicate symbol in $\mathcal{L}$.

Claim 5.5. *Assume that $c_1, \ldots, c_n, d_1, \ldots, d_n \in \mathcal{C}$, and $c_i = d_i$ for all $i = \{1, \ldots, n\}$. Then, $R(\bar{c}) \in T$ iff $R(\bar{d}) \in T$.*

Proof. If $R(\bar{c}) \in T$ then $R(\bar{d})$ is implied by that, and because of 5.2, $R(\bar{d}) \in T$. ■

Let us interpret R as $R^{\mathcal{M}} = \{c_1^*, \ldots, c_n^* : R(c_1, \ldots, c_n) \in T\}$. By the previous claim, $R^{\mathcal{M}}$ is unambiguous. Let us say f is an n-ary function symbol in $\mathcal{L}$ and $c_1, \ldots, c_n \in \mathcal{C}$. Since $\models \exists v f(c_1, \ldots, c_n) = v$ (if not, f would not have an image), and T has the witness property, by 5.2, there exists $c_{n+1} \in \mathcal{C}$ such that $f(c_1, \ldots, c_n) = c_{n+1} \in T$. As above, if $c_i \sim d_i$ for all $i = \{1, \ldots, n+1\}$ then $f(d_1, \ldots, d_n) = d_{n+1} \in T$. Similiarly, if $f(d_1, \ldots, d_n) = d_{n+1} \in T$ and $c_i \sim d_i$ for all $i = \{1, \ldots, n\}$, then $c_{n+1} = d_{n+1}$. So, we have an unambiguous interpretation for the function $f^{\mathcal{M}} : M^n \longrightarrow M$,

$$f^{\mathcal{M}}(c_1^*, \ldots, c_n^*) = d^* \iff f(c_1, \ldots, c_n) = d \in T.$$

We are finished describing the structure $\mathcal{M}$. Now, we will prove by induction that $\mathcal{M} \models T$. First, we show terms behave properly.

Claim 5.6. *Let t be a term with free variables $v_1, \ldots, v_n$, and take $c_1, \ldots, c_n, d \in \mathcal{C}$. Then, $t(c_1, \ldots, c_n) = d \in T$ iff $t^{\mathcal{M}}(c_1^*, \ldots, c_n^*) = d^*$.*

Proof. Suppose $t(c_1, \ldots, c_n) = d$. If t is a constant symbol then $c = d \in T$, and $c^{\mathcal{M}} = c^* = d^*$. If t is the variable v_i, then $c_i = d \in T$ and $t^{\mathcal{M}}(c_1^*, \ldots, c_n^*) = c_i^* = d^*$. Let us assume the claim is true for $t_1, \ldots, t_m$, and t is $f(t_1, \ldots, t_m)$. Since T has the witness property, and by 5.2, it is possible for us to find $d, d_1, \ldots, d_m \in T$ such that $t_i(c_1, \ldots, c_n) = d_i \in T$ and that $f(d_1, \ldots, d_m) = d \in T$ (due to the fact that if there exists a set of free variables satisfying a formula in T, there has to exist a set of constants which also satisfy it in T, and this implies the formula with its variables assigned to the set of constants, which we have seen in 5.2 that is reason enough for this formula to be in T). Therefore, by our assumption, $t_i^{\mathcal{M}}(c_1^*, \ldots, c_n^*) = d_i^*$, and $f^{\mathcal{M}}(d_1^*, \ldots, d_m^*) = d^*$. Hence, $t^{\mathcal{M}}(c_1^*, \ldots, c_n^*) = d^*$, and the induction is complete. To see the converse, we suppose we have $t^{\mathcal{M}}(c_1^*, \ldots, c_n^*) = d^*$. In a similar way than before, we can see that by the witness property and 5.2, we can take $e \in \mathcal{C}$ such that $t(c_1, \ldots, c_n) = e \in T$. But previously we had proven that if $t(c_1, \ldots, c_n) = d$, then $t^{\mathcal{M}}(c_1^*, \ldots, c_n^*) = d^*$, so now we have that $t^{\mathcal{M}}(c_1^*, \ldots, c_n^*) = e^*$. Therefore, $e^* = d^*$, $e = d \in T$. $e = d$ and $t(c_1, \ldots, c_n) = e$ imply $t(c_1, \ldots, c_n) = d$, so by 5.2, $t(c_1, \ldots, c_n) = d \in T$ ■

Finally, we complete the proof via induction on formulas.

Claim 5.7. *For all formulas $\phi(v_1, \ldots, v_n)$ and $c_1, \ldots, c_n \in \mathcal{C}$, $\mathcal{M} \models \phi(\bar{c}^*)$ iff $\phi(\bar{c}) \in T$.*

(1) Atomic formulas
   (i) Say ϕ is $t_1 = t_2$. Because of the witness property an 5.2, we can find d_1 and d_2 constants such that $t_1(\bar{c}) = d_1$ and $t_2(\bar{c}) = d_2$. Also, by 5, $t_1^{\mathcal{M}}(\bar{c}^*) = d_1^*$ and $t_2^{\mathcal{M}}(\bar{c}^*) = d_2^*$. Now,

$$\begin{aligned}\phi(\bar{c}) \in T &\iff t_1(\bar{c}) = t_2(\bar{c}) \\ &\iff d_1 = d_2 \\ &\iff d_1^* = d_2^* \\ &\iff t_1^{\mathcal{M}}(\bar{c}^*) = t_2^{\mathcal{M}}(\bar{c}^*) \\ &\iff \mathcal{M} \models \phi(\bar{c}^*).\end{aligned}$$

   (ii) Say ϕ is $R(t_1, \ldots, t_n)$, then by witness property and 5.2, there exist $d_1, \ldots, d_m \in \mathcal{C}$ such that $t_i(\bar{c}) = d_i$. So,

$$\begin{aligned}\phi(\bar{c}) \in T &\iff R(\bar{d}) \in T \\ &\iff \bar{d}^* \in R^{\mathcal{M}} \\ &\iff \mathcal{M} \models \phi(\bar{c}^*).\end{aligned}$$

(2) Other formulas Assume the claim is true for ψ and β
   (i) Suppose ϕ is $\neg\psi$. If $\mathcal{M} \models \phi(\bar{c}^*)$, then $\mathcal{M} \not\models \psi(\bar{c}^*)$, then $\psi(\bar{c}) \notin T$, and by maximality, $\phi(\bar{c}) \in T$. (This also works inversely)
   (ii) Suppose ϕ is $\psi \longrightarrow \beta$. If $\mathcal{M} \models \phi(\bar{c}^*)$ then either $\mathcal{M} \not\models \psi(\bar{c}^*)$ (we have already seen that case in the previous item) or $\mathcal{M} \models \beta(\bar{c}^*)$, so $\beta(\bar{c}) \in T$. And $\beta \models \phi$, so by 5.2, $\phi(\bar{c}) \in T$. (This also works inversely).
   (iii) If ϕ is $\forall x\psi$, then $\mathcal{M} \models \phi(\bar{c}^*)$ iff $\mathcal{M} \models \psi(d^*, \bar{c}^*)$ for all constants d in $\mathcal{C}$, then $\psi(d, \bar{c}) \in T$, and by 5.2, $\phi \in T$. (Again, this also works inversely).

Thus, induction is finished, and we have fully proven our proposition. ∎

Lemma 5.8. *Let T be a finitely satisfiable $\mathcal{L}$-theory. There is a language $\mathcal{L}^* \supseteq \mathcal{L}$ and a finitely satisfiable $\mathcal{L}$-theory $T^* \supseteq T$ such that any $\mathcal{L}^*$-theory extending T^* has the witness property. We can choose $\mathcal{L}^*$ such that $\mathcal{L}^* = \mathcal{L} + \aleph_0$.*

($\aleph_0$ denotes the cardinality of natural numbers, check the appendix!).

Proof. Let us build a language $\mathcal{L}_1 \supseteq \mathcal{L}$ and a theory $T_1 \supseteq T$ in the following manner: for any $\mathcal{L}$-formula $\phi(v)$, we will have a constant symbol in $\mathcal{L}_1$ such that $T_1 \models \exists v\phi(v) \longrightarrow \phi(c)$. For an $\mathcal{L}$-formula ϕ, denote this constant symbol as c_ϕ, so that $\mathcal{L}_1$ is $\mathcal{L} \cup \{c_\phi : \phi$ an $\mathcal{L}$-formula$\}$. For each $\mathcal{L}$-formula $\phi(v)$, Θ_ϕ will be the $\mathcal{L}_1$-sentence $\exists v\phi(v) \longrightarrow \phi(c_\phi)$. T_1 will be $T \cup \{\Theta_\phi : \phi(v)$ an $\mathcal{L}$-formula$\}$. With this, we will be always be able to extend T so it has the witness property. We now have to prove that every time we extend T, it remains finitely satisfiable, so as to finally prove our statement.

Claim 5.9. *T_1 is finitely satisfiable.*

Proof. Assume there exists a counterexample, where $\Delta_0 \cup \{\Theta_{\phi_1}, \ldots, \Theta_{\phi_n}\}$ is a non satisfiable subset of T_1, and Δ_0 is a finite subset of T. In the case of Δ_0, there exists an $\mathcal{L}$-structure $\mathcal{M}$ such that $\mathcal{M} \models \Delta_0$. Make an $\mathcal{L} \cup (c_{\phi_1}, \ldots, c_{\phi_n})$-structure $\mathcal{M}'$ from $\mathcal{M}$, so that all interpretations remain the same, only we added more constant symbols. Since the interpretation remains the same, $\mathcal{M}' \models \Delta_0$. How do we make all interpretations remain the same? We show how $(c_{\phi_1}, \dots, c_{\phi_n})$ are interpreted in our new model. If $\mathcal{M} \models \phi_i(v)$, we choose a_i in M such that $\mathcal{M} \models \phi_i(a_i)$, and we say $c_{\phi_i}^{\mathcal{M}'} = a_i$. If $\mathcal{M} \not\models \phi_i(v)$, then $c_{\phi_i}^{\mathcal{M}'}$ can be any element of M. Since $\mathcal{M}' \models \Delta_0$ and $\mathcal{M}' \models \Theta_{\phi_i}$ (this is clear from what what we just stated and the fact that T_1 has the witness property). Then $\Delta_0 \cup \{\Theta_{\phi_1}, \dots, \Theta_{\phi_n}\}$ is satisfied by $\mathcal{M}'$. Contradiction! ■

We keep on constructing like this, so we have $\mathcal{L} \subseteq \mathcal{L}_1 \subseteq \mathcal{L}_2 \dots$ and $T \subseteq T_1 \subseteq T_2 \dots$ Then, if $\phi(v)$ is an $\mathcal{L}_i$-formula, then there exists a constant symbol $c \in \mathcal{L}_{i+1}$ such that $T_{i+1} \models \exists v \phi(v) \longrightarrow \phi(c)$. Let $\mathcal{L}^*$ be $\bigcup \mathcal{L}_i$ and T^*, $\bigcup T_i$. T^* has the witness property (we built it to have it), so if Δ is a finite subset of T^*, then $\Delta \in T_i$. Every theory T_i is finitely satisfiable, so Δ is satisfiable, and thus, T^* is finitely satisfiable. Now, we know that in any countable $\mathcal{L}$-language, the cardinality set of all formulas we can make is $|\mathcal{L}| + \aleph_0$, since we can combine symbols in countably many ways ($\mathcal{L}$ being the set of all constants, functions and relation symbols). Since the cardinality of the union of bounded sets is equal to the cardinality of the biggest set, then we can say $|\mathcal{L}^*| = |\mathcal{L}| + \aleph_0$. ■

Lemma 5.10. *Let T be a finitely satisfiable $\mathcal{L}$-theory and ϕ an $\mathcal{L}$-sentence. Then, either $T \cup \phi$ or $T \cup \neg\phi$ is finitely satisfiable.*

Proof. Suppose this is not true. Then, there must exist finite subsets R and S of T such that $R \cup \phi$ and $S \cup \neg\phi$ are both unsatisfiable. If $R \cup \phi$ is unsatisfiable, this means every model of R is not a model of ϕ, thus, $R \models \neg\phi$. Since every model of R satisfies $\neg\phi$, and $R \cup S$ is satisfiable (so it has a model), then there exists a model of S which is also a model of $\neg\phi$, meaning $S \cup \neg\phi$ is satisfiable. Contradiction! ■

Corollary 5.11. *If T is a finitely satisfiable $\mathcal{L}$-theory, then there is a maximal finitely satisfiable $\mathcal{L}$-theory $T^* \supseteq T$.*

Proof. Let I be the set of all the finitely satisfiable $\mathcal{L}$-theories containing T. Define a partial ordering in I by inclusion (meaning, elements of the set are ordered by which one of them is a subset of which). Let $C \in I$ be a *chain* (a subset of the set of subsets of I, where for all subsets X, Y, either $X \subseteq Y$ or $Y \subseteq X$). Let $T_C = \bigcup\{\Sigma : \Sigma \in C\}$. For Δ a subset of T_C, we know by our definition of chain that $\Delta \subseteq \Sigma$ for some $\Sigma \in T$. Every Σ is finitely satisfiable by assumption (it is one of the finitely satisfiable $\mathcal{L}$-theories in I). Then, Δ is satisfiable, so T_C is finitely satisfiable.

This means every chain in I has to have an upper bound, so we can use Zorn's Lemma (in the appendix) to find $T^{'}$ maximal in the partial ordering. If $T^{'}$ is not maximal, then for a formula ϕ, $T^{'} \cup \phi$ is not finitely satisfiable. But, by 5.10, this means $T^{'} \cup \neg\phi$ is finitely satisfiable, and $T^{'}$ is not maximal in the partial ordering. Contradiction! ■

After all this work, we are able to prove this version of the Compactness Theorem, which will allow us to keep advancing towards abstract algebra.

Theorem 5.12. *If T is a finitely satisfiable $\mathcal{L}$-theory and κ is an infinite cardinal with $\kappa \geq |\mathcal{L}|$, then there is a model of T of cardinality at most κ.*

Proof. By 5.8, we are always able to find $\mathcal{L}^* \supseteq \mathcal{L}$ and a finitely satisfiable $\mathcal{L}^*$-theory $T^* \supseteq T$ such that any $\mathcal{L}^*$-theory extending T^* has the witness property and cardinality at most κ (since κ is infinite). Then, by 5.11, we can find a maximal finitely satisfiable $\mathcal{L}$-theory $T^{'} \supseteq T^*$. Since $T^{'}$ has the witness property, then 5.3 tells us that there is $\mathcal{M} \models T$ with $|M| \leq \kappa$. ■

6. The Theory of ACF_p

Definition 6.1. An $\mathcal{L}$-theory T is *complete* iff for any $\mathcal{L}$-sentence ϕ, either $T \models \phi$ or $T \models \neg\phi$.

Proposition 6.2. *Let T be an $\mathcal{L}$-theory with infinite models. If κ is an infinite cardinal and $\kappa \geq |\mathcal{L}|$, then there is a model of T of cardinality κ.*

Proof. Let $\mathcal{L}^*$ be $\mathcal{L} \cup \{c_\alpha : \alpha < \kappa\}$, where each c_α is a new constant symbol. Define T^* as $T \cup \{c_\alpha \neq c_\beta : \alpha, \beta < \kappa, \alpha \neq \beta\}$. First we will show that T^* is finitely satisfiable. Take Δ a finite subset of T^*, so $\Delta \subseteq T \cup \{c_\alpha \neq c_\beta : \alpha, \beta \in I, \alpha \neq \beta\}$, where I is a finite subset of κ. Let $\mathcal{M}$ be an infinite model of T. Then, for every constant $c_\alpha \in I$, we can interpret it as a different element of M. This means these sentences are satisfied by $\mathcal{M}$, so $\mathcal{M} \models \Delta$. This means T^* is finitely satisfiable.

Now, we can see that since that since T^* has κ constants, every model of T^* has a cardinality of at least κ. Since $T \subseteq T^*$, every model of T^* has to satisfy T. We apply 5.12 and take a model $\mathcal{M}$ of T^* of cardinality at most κ. Then, $|M| = \kappa$, and by what we said before, $\mathcal{M} \models T$. ■

Definition 6.3. Let κ be an infinite cardinal and T a theory with models of cardinality κ. If any two models of T with cardinality κ are isomorphic, we say T is κ-categorical.

The following proposition will regard topics from abstract algebra. We recommend the reader to read first the appendix (if they are not confident on their knowledge over this subject).

Proposition 6.4. *The theory ACF_p is κ-categorical for all uncountable cardinals κ.*

Proof. Two algebraically closed fields are isomorphic if and only if they have the same transcendence degree and characteristic. The cardinality of an algebraically closed field of transcendence degree γ is $\gamma + \aleph_0$. If $\kappa > \aleph_0$, an algebraically closed field of cardinality κ also has transendence degree κ. Then, any two algebraically closed fields of the same characteristic and same uncountable cardinality are isomorphic, then their models also are isomorphic. ■

Theorem 6.5 (Vaught's Test)**.** *Let T be a satisfiable $\mathcal{L}$-theory with no finite models that is κ-categorical for some infinite cardinal $\kappa \geq |\mathcal{L}|$. Then T is complete.*

Proof. Suppose T is not complete. Then, there exists a sentence ϕ such that $T \not\models \phi$ and $T \not\models \neg\phi$. This means that theories $T \cup \phi$ and $T \cup \neg\phi$ are satisfiable. Since all models of T are infinite, then the models of these theories also are infinite. By 6.2, there exists a model of each with cardinality κ, let us call them $\mathcal{M}$ and $\mathcal{N}$, respectively. Since they are models of T with cardinality κ, they are isomorphic. By 3.4, this means they are elementary equivalent, but since ϕ is true in one and false in the other, we have a contradiction. ■

Corollary 6.6. *ACF_p is complete.*

Proof. By our two previous statements, this proof is immediate. ACF_p is κ-categorical. Since it has no finite models (and it has uncountable models), then it is complete. ■

Corollary 6.7 (Lefschetz Principle)**.** *Let ϕ be a sentence in the language of rings. The following statements are equivalent:*

(i) *ϕ is true in the complex numbers.*
(ii) *ϕ is true in some algebraically closed field of characteristic* 0.
(iii) *ϕ is true in every algebraically closed field of characteristic* 0.

(iv) *There are arbitrarily large primes p such that ϕ is true in some algebraically closed field of characteristic p.*
(v) *There exists m such that ϕ is true for all ACF_p, with $p > m$.*

We will begin by proving (i)-(iii) are equivalent, and then see that (ii) implies (v), (v) implies (iv) and (iv) implies (ii).

Proof. If ϕ is true in some algebraically closed field of characteristic 0 and false in another, then $ACF_0 \not\models \phi$ and $ACF_0 \not\models \neg\phi$. But, since ACF_0 is complete, then this is impossible. Also, complex numbers are an algebraically closed field of characteristic 0. Thus, we see (i)-(iii) are equivalent. It is also evident that (v) implies (iv).

Suppose $ACF_0 \models \phi$. Then by 4.8, there exists a finite theory $\Delta \subset ACF_0$ such that $\Delta \models \phi$. Since Δ is finite, then it can only have finitely many sentences $\neg\psi_p$ (see the appendix for this). Thus, for large enough p, $ACF_p \models \Delta$, and thus $ACF_p \models \phi$.

Lastly, we can see that (iv) implies (ii) by contraposition. Suppose $ACF_0 \not\models \phi$. By the completness of ACF_0, $ACF_0 \models \neg\phi$. By the argument above, $ACF_p \models \neg\phi$ for sufficiently large ϕ, so (iv) fails. ∎

7. The Ax-Grothendieck Theorem

Theorem 7.1 (Ax-Grothendieck Theorem)**.** *Let $f : \mathbb{C}^n \longrightarrow \mathbb{C}^n$ be an injective polynomial. Then, f is surjective.*

For the proof, we use the fact that every injective function from a finite set to another with the same cardinality is surjective. This applies to finite fields. We can build a first order sentence in the language of rings that holds in a field k if and only if every injective polynomial $h : k^n \longrightarrow k^n$ is surjective. After that, we prove that the sentence also holds for any increasing union of finite fields, specifically, the algebraic closure of a finite field. Then, by the previous corollary, the sentence holds in the complex numbers.

Proof. Note that the statement is trivial for polynomials $f : k^n \longrightarrow k^n$, where k is a finite field. If f is injective, f is neccesarily one-to-one because k is finite, so if there was $a \in k$ such that a was not in the range, then at least two elements in the domain of the function have the same image.

Now, assume the theorem is false, and that $f(X)$, is a counterexample, with $X = (X_1, \ldots, X_n)$, and $f(X) = (f_1(X_1), \ldots, f_n(X_n))$. Suppose every $f_i \in \mathbb{C}[X]$ has degree at most d. We can build an $\mathcal{L}_r$-sentence $\phi_{n,d}$, such that for K a field, $K \models \phi_{n,d}$ if and only if the theorem is true for all polynomials $K^n \longrightarrow K^n$ with degree at most d. To do so, we quantify over polynomials of degree at most d by quantifying over the coefficients, in the following manner.

We start by writing a sentence $\alpha_{(i_1,\ldots,i_n)}$ which describes an n-variable polynomial with coefficients $a_{(i_1,\ldots,i_n)}$ which is injective, where i_k indicates the exponent of the variable multiplied by a_{i_k} and all exponents are less than or equal to d. Since $f = (f_1, \ldots, f_n)$, then we will write $a_{k,(i_1,\ldots,i_n)}$ to denote that $a_{(i_1,\ldots,i_n)}$ is a coefficient of k. Then, the sentence would look like this:

$$\forall x_1,\dots,\forall x_n \forall y_1,\dots,\forall y_n \left(\left(\bigwedge_{k\le n}\sum_{(i_1,\dots,i_n)} a_{k,(i_1,\dots,i_n)} x_1^{i_1}\cdots x_n^{i_n} = \sum_{(i_1,\dots,i_n)} a_{k,(i_1,\dots,i_n)} y_1^{i_1}\cdots y_n^{i_n}\right) \longrightarrow \bigwedge_{i=1,\dots,n} x_i = y_i\right).$$

We also build the sentence $\beta_{i_1,\dots,i_n}$, which says such polynomial is surjective. This would look like this:

$$\forall z_1,\dots,\forall z_n \exists x_1,\dots,x_n \left(\bigwedge_{k\le n}\sum_{(i_1,\dots,i_n)} a_{k,(i_1,\dots,i_n)} x_1^{i_1}\cdots x_n^{i_n} = z_k\right).$$

Now, we quantify over the coefficients and define $\phi_{n,d}$ to be the following sentence:

$$\underset{k,(i_1,\dots,(i_n)}{\forall} a_{k,(i_1,\dots,i_n)}(\alpha_{i_1,\dots,i_n} \longrightarrow \beta_{i_1,\dots,i_n}).$$

Therefore, it is clear we can translate this theorem to a first order language.For example, $\phi_{2,2}$ would look like this:

$$\forall a_{1,(0,0)} \forall a_{1,(0,1)} \forall a_{1,(0,2)} \forall a_{1,(1,0)} \forall a_{1,(1,1)} \forall a_{1,(2,0)} \forall a_{2,(0,0)} \forall a_{2,(0,1)} \forall a_{2,(0,2)} \forall a_{2,(1,0)} \forall a_{2,(1,1)} \forall a_{2,(2,0)}$$
$$\left[\forall x_1 \forall x_2 \forall y_1 \forall y_2 \left(\left(\sum a_{1,(i_1,i_2)} x_1^{i_1} x_2^{i_2} = \sum a_{1,(i_1,i_2)} y_1^{i_1} y_2^{i_2} \wedge \sum a_{2,(i_1,i_2)} x_1^{i_1} x_2^{i_2} = \sum a_{2,(i_1,i_2)} y_1^{i_1} y_2^{i_2}\right)\right.\right.$$
$$\left.\left.\longrightarrow x_1 = y_1 \wedge x_2 = y_2\right) \longrightarrow \forall z_1 \forall z_2 \exists x_1 \exists x_2 \left(\sum a_{1,(i_1,i_2)} x_1^{i_1} x_2^{i_2} = z_1 \wedge \sum a_{2,(i_1,i_2)} x_1^{i_1} x_2^{i_2} = z_2\right)\right].$$

Claim 7.2. *The sentence is true in some algebraically closed field with characteristic p, with arbitrarilly large p.*

Proof. Suppose not. We know that if the polynomial $f : k^n \longrightarrow k^n$ is injective then it is bijective (k is a finite field). Let $\mathbb{F}_p^{alg}$ be the algebraic closure of the finite field with p elements. Let us assume that there exists a counterexample, $f : \mathbb{F}_p^{alg} \longrightarrow \mathbb{F}_p^{alg}$ which is injective but not surjective. Let $\bar{a} \in \mathbb{F}_p^{alg}$ be the coefficients of f and $(b_1,\dots,b_n) \in \mathbb{F}_p^{alg}$ an element not in the range of f. Consider the subfield $K \in \mathbb{F}_p^{alg}$ generated by $\bar{a}$ and $(b_1,\dots,b_n)$. It follows that $f\restriction_{K^n}$ (the function f, with its domain restricted to K^n) is an injective but not surjective map from K^n into itself (given how K is a field generated by $\bar{a}$ and $(b_1,\dots,b_n)$). Since $\mathbb{F}_p^{alg} = \bigcup_{n=1}^{\infty} \mathbb{F}_{p^n}$, then any finitely generated subfield is contained in some finite sub-union $\bigcup_{n=1}^{N} \mathbb{F}_{p^n}$. Thus, K is always finite, so we have a contradiction. Then, $\mathbb{F}_p^{alg} \models \phi_{n,d}$ for all primes p, so this is true in an algebraically closed field with characteristic p, for an arbitrarily large p. ■

By the claim, we see that for $\phi_{n,d}$, the statement (iv) in 6.7 is true. Then, the statement (i) is also true, and $\mathbb{C} \models \phi_{n,d}$. ■

8. Non-Standard Models

Within model theory, whenever we have a theory and a model, we can find another model of that theory, non-isomorphic to the original one. This new model, called the *non-standard* model of the theory, may help us to prove some things for that theory. We will not be able to prove things that were impossible to prove in the standard model, but the procedure might actually become easier. In this section we will present the Löwenheim-Skolem Theorems, which guarantee us the existence of non-standard models (for infinite models). These theorems consist in two parts: the Upward part, proving the existence of larger non-standard models, and the Downward part, doing the same but for smaller models.

Definition 8.1. Let us have two $\mathcal{L}$-structures $\mathcal{M}$ and $\mathcal{N}$, and let $h : M \longrightarrow N$ be an $\mathcal{L}$-embedding. If for every $\mathcal{L}$-formula $\phi(\bar{v})$ and all $(\bar{a}) \in M$,

$$\mathcal{M} \models \phi(\bar{a}) \iff \mathcal{N} \models \phi(h(\bar{a}))$$

then we say h is an *elementary embedding.* If $M \subset N$ we say $\mathcal{M}$ is the *elementary substructure* of $\mathcal{N}$ (and $\mathcal{N}$ is the *elementary extension* of $\mathcal{M}$).

Definition 8.2. Let $\mathcal{M}$ be an $\mathcal{L}$-structure, and let us add to the language a constant m for every element of M, so as to build the language $\mathcal{L}_M$.

The *atomic diagram* of $\mathcal{M}$ will be $\{\phi(\bar{m}) : \mathcal{M} \models \phi$ and ϕ is an atomic $\mathcal{L}$-formula or its negation$\}$.

The *elementary diagram* of $\mathcal{M}$ will be $\{\phi(\bar{m}) : \mathcal{M} \models \phi$ and ϕ is an $\mathcal{L}$-formula$\}$.

We will denote these as Diag($\mathcal{M}$) and ElDiag($\mathcal{M}$), respectively.

Lemma 8.3. *Let $\mathcal{N}$ be an $\mathcal{L}_M$-structure.*

(i) *If $\mathcal{N} \models$Diag($\mathcal{M}$) and we view $\mathcal{N}$ as an $\mathcal{L}$-structure, then there is an $\mathcal{L}$-embedding $M \longrightarrow N$.*
(ii) *If $\mathcal{N} \models$ElDiag($\mathcal{M}$), then there is an elementary embedding $M \longrightarrow N$.*

Proof. Let us see each case separately:

(i) Consider the function $j : M \longrightarrow N$, for which $j(m) = m^{\mathcal{N}}$, meaning $j(m)$ is the interpretation in $\mathcal{N}$ for each constant in $\mathcal{L}_M$. For m_1, m_2 distinct elements in Diag($\mathcal{M}$), we have $m_1 \neq m_2 \in$Diag($\mathcal{M}$), so $j(m_1) \neq j(m_2)$. Now with functions, if we have $f^{\mathcal{M}}(\bar{m}) = m_i$, then $f(\bar{m}) = m_i$ is a formula in Diag($\mathcal{M}$) and $f^{\mathcal{N}}(j(\bar{m}) = j(m_i)$. Let R be a relation symbol, so for $(\bar{m}) \in R^{\mathcal{M}}$, then $R(\bar{m}) \in$Diag($\mathcal{M}$) and $(j(\bar{a})) \in R^{\mathcal{N}}$. Since we see j meets all the conditions, then it is an $\mathcal{L}$-embedding.
(ii) The function j from above is elementary, we can see that by repeating the same argument and replacing Diag($\mathcal{M}$) with ElDiag($\mathcal{M}$).

■

Theorem 8.4 (Upward Löwenheim-Skolem Theorem)**.** *Let $\mathcal{M}$ be an infinite $\mathcal{L}$-structure and let κ be a cardinal number such that $\kappa \geq |M| + |\mathcal{L}|$. Then, there exists an elementary extension $\mathcal{N}$ such that $|N| = \kappa$.*

Proof. Since $\mathcal{M} \models$ElDiag($\mathcal{M}$), then ElDiag($\mathcal{M}$) is satisfiable. We can see that by 5.12 we can find a model $\mathcal{N}$ of DiagEl($\mathcal{M}$) of cardinality at most $|\mathcal{L}_M| = |\mathcal{L}| + |M| + \kappa = \kappa$.

Now, we know that if we have $\epsilon, \omega < \kappa$, then $\mathcal{N} \models m_\epsilon \neq m_\omega$. This means $m_\epsilon \neq m_\omega$, so the function from ϵ to $m_\epsilon^{\mathcal{N}}$ is an injective embedding from κ to N. Therefore, $|N| \geq \kappa$. Along with the previous result we get $|N| = \kappa$. Then, by lemma 8.3, there exists an elementary embedding $M \longrightarrow N$. Since $\mathcal{N}$ is a model of the elementary diagram of $\mathcal{M}$, then $M \subset N$ and $\mathcal{N}$ is an elementary extension. ∎

Proposition 8.5 (Tarski-Vaught Test)**.** *Suppose that $\mathcal{M}$ is a substructure of $\mathcal{N}$. Then, $\mathcal{M}$ is an elementary substructure if and only if, for any formula $\phi(v, \bar{w})$ and $\bar{a} \in M$, if there is $b \in N$ such that $\mathcal{N} \models \phi(b, \bar{a})$, then there is $c \in M$ such that $\mathcal{N} \models \phi(c, \bar{a})$.*

Proof. If $\mathcal{M}$ is an elementary substructure of $\mathcal{N}$, then the following part is clearly true.

Let us see the converse. We must show that for all $\bar{a} \in M$ and all formulas $\psi(\bar{v})$

$$\mathcal{M} \models \psi(\bar{a}) \iff \mathcal{N} \models \psi(\bar{a}).$$

We will prove this by induction on formulas.

Claim 8.6. *If $\mathcal{M}$ is a substructure of $\mathcal{N}$, $\bar{a} \in M$, and $\phi(\bar{v})$ is a quantifier free formula, then $\mathcal{M} \models \phi(\bar{a}) \iff \mathcal{N} \models \phi(\bar{a})$.*

Proof. We first show terms behave well. By induction on terms, we see that if $t(\bar{v})$ is a term and $\bar{b} \in M$, then $t^{\mathcal{M}}(\bar{b}) = t^{\mathcal{N}}(\bar{b})$.

If t is a constant symbol c, then $c^{\mathcal{M}} = c^{\mathcal{M}}$.

If t is the variable b, then $t^{\mathcal{M}}(\bar{b}) = b = t^{\mathcal{N}}(\bar{b})$.

If $t = f(t_1, \ldots, t_n)$ and $t_i^{\mathcal{M}}(\bar{b}) = t_i^{\mathcal{N}}(\bar{b})$. Since $M \subseteq N$, $f^{\mathcal{M}} = f^{\mathcal{N}}|M^n$. Then,

$$\begin{aligned} t^{\mathcal{M}}(\bar{b}) &= f^{\mathcal{M}}(t_1^{\mathcal{M}}(\bar{b}), \ldots, t_n^{\mathcal{M}}(\bar{b})) \\ &= f^{\mathcal{M}}(t_1^{\mathcal{M}}(\bar{b}), \ldots, t_n^{\mathcal{M}}(\bar{b})) \\ &= f^{\mathcal{N}}(t_1^{\mathcal{N}}(\bar{b}), \ldots, t_n^{\mathcal{N}}(\bar{b})) \\ &= t^{\mathcal{N}}(\bar{b}). \end{aligned}$$

Now, we prove the claim by induction on formulas. If ϕ is $t_1 = t_2$, then

$$\mathcal{M} \models \phi(\bar{a}) \iff t_1^{\mathcal{M}} = t_2^{\mathcal{M}} \iff t_1^{\mathcal{N}} = t_2^{\mathcal{N}} \iff \mathcal{N} \models \phi(\bar{a}).$$

If ϕ is $R(t_1, \ldots, t_n)$, then

$$\begin{aligned} \mathcal{M} \models \phi(\bar{a}) &\iff (t_1^{\mathcal{M}}(\bar{a}), \ldots, t_n^{\mathcal{M}}(\bar{a})) \in R^{\mathcal{M}} \\ &\iff (t_1^{\mathcal{M}}(\bar{a}), \ldots, t_n^{\mathcal{M}}(\bar{a})) \in R^{\mathcal{N}} \\ &\iff (t_1^{\mathcal{N}}(\bar{a}), \ldots, t_n^{\mathcal{N}}(\bar{a})) \in R^{\mathcal{N}} \\ &\iff \mathcal{N} \models \phi(\bar{a}). \end{aligned}$$

Thus, our claim is proved for all atomic formulas. Assume the claim is true for ψ and β. Then, suppose ϕ is $\neg\psi$. We have

$$\mathcal{M} \models \phi(\bar{a}) \iff \mathcal{M} \not\models \psi(\bar{a}) \iff \mathcal{N} \not\models \psi(\bar{a}) \iff \mathcal{N} \models \phi(\bar{a}).$$

Lastly, let ϕ be $\psi \longrightarrow \beta$. Then,

$$\mathcal{M} \models \phi(\bar{a}) \iff \mathcal{M} \models \neg\psi(\bar{a}) \iff \mathcal{N} \models \neg\psi(\bar{a}) \iff \mathcal{N} \models \phi(\bar{a})$$

or

$$\mathcal{M} \models \phi(\bar{a}) \iff \mathcal{M} \models \beta(\bar{a}) \iff \mathcal{N} \models \beta(\bar{a}) \iff \mathcal{N} \models \phi(\bar{a}).$$

∎

By the claim, we know that if $\phi(\bar{v})$ is quantifier free, then $\mathcal{M} \models \phi(\bar{a})$ if and only if $\mathcal{N} \models \phi(\bar{a})$. Therefore, our statement is true for all atomic formulas.

If the statement is true for ψ, then

$$\mathcal{M} \models \neg\psi(\bar{a}) \iff \mathcal{M} \not\models \psi(\bar{a}) \iff \mathcal{N} \not\models \psi(\bar{a}) \iff \mathcal{N} \models \neg\psi(\bar{a}).$$

If the statement is true for $\psi \longrightarrow \beta$, then

$$\mathcal{M} \models (\psi \longrightarrow \beta)(\bar{a}) \iff \mathcal{M} \models \neg\psi(\bar{a}) \iff \mathcal{N} \models \neg\psi(\bar{a}) \iff \mathcal{N} \models (\psi \longrightarrow \beta)(\bar{a})$$

or

$$\mathcal{M} \models (\psi \longrightarrow \beta)(\bar{a}) \iff \mathcal{M} \models \beta(\bar{a}) \iff \mathcal{N} \models \beta(\bar{a}) \iff \mathcal{N} \models (\psi \longrightarrow \beta)(\bar{a}).$$

Now, suppose the claim is true for $\psi(v, \bar{w})$. Let $\bar{a} \in M$. If $\mathcal{M} \models \forall v\psi(v, \bar{a})$ then for all $b \in M$, $\mathcal{M} \models \psi(b, \bar{a})$. By our inductive assumption, $\mathcal{N} \models \psi(b, \bar{a})$, so $\mathcal{N} \models \forall v\psi(v, \bar{a})$.

If we want to see the converse, then assume $\mathcal{N} \models \forall v\psi(v, \bar{a})$. This means that for all $b \in M$, $\mathcal{N} \models \psi(b, \bar{a})$. By inductive hypothesis, $\mathcal{M} \models \psi(b, \bar{a})$, so $\mathcal{M} \models \forall v\psi(v, \bar{a})$. ■

Definition 8.7. An $\mathcal{L}$-theory T has *built-in Skolem functions* if for all $\mathcal{L}$-formulas $\phi(v, \bar{w})$ there exists a function symbol f such that $T \models \forall\bar{w}((\exists v\phi(v, \bar{w})) \longrightarrow (\phi(f(\bar{w}), \bar{w}))$.

Lemma 8.8. *Let T be an $\mathcal{L}$-theory. There are $\mathcal{L}^* \supseteq \mathcal{L}$ and $T^* \supseteq T$ an $\mathcal{L}^*$-theory such that T^* has built-in Skolem functions, and if $\mathcal{M} \models T$, then we can expand $\mathcal{M}$ to $\mathcal{M}^* \models T^*$. We can choose $\mathcal{L}^*$ such that $|\mathcal{L}^*| = |\mathcal{L}| + \aleph_0$.*

Proof. We build a sequence of languages $\mathcal{L} = \mathcal{L}_0 \cup \mathcal{L}_1, \ldots$ and a sequence of $\mathcal{L}_i$-theories $T = T_0 \cup T_1, \ldots$. Given $\mathcal{L}_i$, let $\mathcal{L}_{i+1} = \mathcal{L}_i \cup \{f_\phi : \phi(v, w_1, \ldots, w_n) \text{ an } \mathcal{L}_i\text{-formula}\}$, where f_ϕ is an n-ary function symbol. For $\phi(v, \bar{w})$ an $\mathcal{L}_i$-formula, let Ψ_ϕ be the sentence

$$\forall\bar{w}((\exists v\phi(v, \bar{w})) \longrightarrow (\phi(f_\phi(\bar{w}), \bar{w}))$$

and let $T_{i+1} = T_i \cup \{\Psi_\phi : \phi \text{ an } \mathcal{L}_i \text{ formula}\}$.

Claim 8.9. *If $\mathcal{M} \models T_i$, then we can interpret the function symbols of $\mathcal{L}_{i+1} \setminus \mathcal{L}_i$ so that $\mathcal{M} \models T_{i+1}$.*

Proof. Let c be some fixed element of M. If $\phi(v, w_1, \ldots, w_n)$ is an $\mathcal{L}_i$-formula, we find a function $g : M^n \longrightarrow M$ such that if $\bar{a} \in M^n$ and $X_{\bar{a}} = \{b \in M : \mathcal{M} \models \phi(b, \bar{a})\}$ is nonempty, then $g(\bar{a}) \in X_a$, and if $X_{\bar{a}} = \emptyset$, then $g(\bar{a}) = c$ (the choice in this case is irrelevant). Thus, if $\mathcal{M} \models \exists v\phi(v, \bar{a})$, then $\mathcal{M} \models \phi(g(\bar{a}), \bar{a})$. If we interpret f_ϕ as g, then $\mathcal{M} \models \Psi_\phi$. ■

Let $\mathcal{L}^* = \bigcup \mathcal{L}_i$ and $T^* = \bigcup T_i$. If $\phi(v, \bar{w})$ is an $\mathcal{L}^*$-formula, then $\phi \in \mathcal{L}_i$ for some i and $\Psi_\phi \in T_{i+1} \subseteq T^*$, so T^* has built-in Skolem functions. By iterating the claim, we see that for any $\mathcal{M} \models T$ we can interpret the symbols of$\mathcal{L}^* \setminus \mathcal{L}$ to make $\mathcal{M} \models T^*$. Because we have added one function symbol to $\mathcal{L}_{i+1}$ for each $\mathcal{L}_i$-formula, $|\mathcal{L}_{i+1}| = |\mathcal{L}_i| + \aleph_0$ so $|\mathcal{L}^*|$ has the desired cardinality. ■

Theorem 8.10 (Downward Löwenheim-Skolem Theorem)**.** *Let $\mathcal{M}$ be a model in some language $\mathcal{L}$. Then for any subset $X \subseteq M$, there exists an elementary substructure $\mathcal{N}$ containing X, with $|N| \leq |X| + |\mathcal{L}| + \aleph_0$. In particular, taking X to be an arbitrary subset of size κ with $|L| \leq \kappa \leq |M|$, we can find an elementary substructure of $\mathcal{M}$ of size κ.*

Proof. By 8, we may assume that $\mathrm{Th}(\mathcal{M})$ (the set of all formulas true in $\mathcal{M}$) has built in Skolem functions. Let $X_0 = X$. Given X_i, $X_{i+1} = X_i \cup \{f^{\mathcal{M}}(\bar{a}) : f$ an n-ary function symbol, $\bar{a} \in X_i^n\}$. Let N be $\bigcup X_i$, then $|N| \leq |X| + |\mathcal{L}| + \aleph_0$ (check the appendix).

If f is an n-ary function symbol of $\mathcal{L}$ and $\bar{a} \in N^n$, then $\bar{a} \in X_i^n\}$ for some i and $f^{\mathcal{M}}(\bar{a}) \in X_{i+1}^n \subseteq N$. Therefore, $f^{\mathcal{M}}|N : N^n \longrightarrow N$. Thus, we are able to interpret f as $f^{\mathcal{M}}|N = f^{\mathcal{N}}$. For R an n-relation symbol, let $R^{\mathcal{N}} = R^{\mathcal{N}} \cap N^n$. Let c be a constant in $\mathcal{L}$, then there exists a Skolem function $f \in \mathcal{L}$ such that $f(x) = c^{\mathcal{M}}$ for all $x \in M$. Then, $c^{\mathcal{M}} \in N$. Let $c^{\mathcal{N}} = c^{\mathcal{M}}$. This makes $\mathcal{N}$ an $\mathcal{L}$-structure, substructure of $\mathcal{M}$.

For $\phi(v, \bar{w})$ a formula, $\bar{a}, b \in M$, and $\mathcal{M} \models \phi(b, \bar{a})$, then $\mathcal{M} \models \phi(f(\bar{a}), \bar{a})$ for some function symbol $f \in \mathcal{L}$. By construction, $f^{\mathcal{M}}(\bar{a}) \in N$. So, by Tarski-Vaught Test, $\mathcal{N}$ is an elementary substructure of $\mathcal{M}$. ■

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9. Appendix

9.1. **Induction on Formulas.** A powerful tool to prove things in model theory is induction on formulas. It was used several times in this paper, since it is very useful to prove that some property is met by all formulas. It works in the following way.

First, we prove our base case. We show that the property we want to prove is met by all atomic formulas. Then, we make our induction hypothesis, and suppose the property applies for formulas ψ and β. Afterwards, we prove that if our hypothesis is true, then the property is met by all formulas, by proving it is met by formulas $\neg\psi$, $\psi \longrightarrow \beta$ and $\forall x \psi$. If this is true, then the property is met by all formulas, and induction is completed. But, why does this work?

This is due to the fact that all logical connectives are replaceable by just $\{\longrightarrow, \neg\}$. It is clear that for formulas $\{\alpha, \beta\}$, the formula $\alpha \vee \beta$ is equivalent to $(\neg\alpha) \longrightarrow \beta$, and $\alpha \wedge \beta$ is equivalent to $\neg(\alpha \longrightarrow (\neg\beta))$. We can make $\alpha \longleftrightarrow \beta$ by $(\alpha \longrightarrow \beta) \wedge (\beta \longrightarrow \alpha)$ since we have already proven $\wedge$ is replaceable. For the quantifiers, it is enough to see that $\exists x \alpha$ is the same as saying $\neg\forall x \neg\alpha$. Then this three symbols are enough to build all formulas.

9.2. Cardinals.

Definition 9.1. The *cardinality* of a set is the quantity of elements contained by it.

For the cardinality of the set A, we will write $|A|$.

Proposition 9.2. *The following statements are equivalent:*

- $|A| = |B|$.
- *There exists a bijection* $f : A \longrightarrow B$.

Definition 9.3. We will say $|A|$ is *countable* if there exists a bijective map $A \longrightarrow \mathbb{N}$. If this is true, then $|A| = \aleph_0$.

Let us say we want to add two infinite cardinals κ and λ. Then, the opperation follows as $\kappa + \lambda = \max\{\kappa, \lambda\}$.

Corollary 9.4. *If* $|I| = \kappa$ *and* $|A_i| \leq \kappa$ *for all* $i \in I$*, then* $|A_i| \leq \kappa$.

9.3. Abstract Algebra.

Even though in this paper we will frequently discuss topics related to abstract algebra, we consider the reader does not need a deep understanding of the subject. Nevertheless, we still think some notions will be useful.

Definition 9.5. A *group* G is a set in which you can perform one operation $*$, and for any $x, y \in G$, $x * y \in G$ (i.e. it is closed). Also, it has an *identity element* e such that $x * e = x$, and an inverse x^{-1} for every $x \in G$ such that $x^{-1} * x = e$.

Definition 9.6. A *ring* is a group which is closed under two operations, and is abelian under one of them (i.e. $x * y = y * x$). It also has the associative and distributive properties.

Definition 9.7. A *field* F is a ring which is abelian under two operations if the identity of one of them is removed from F. Fields can be infinite or finite. In this last case, the cardinality of it will be in the form p^n, meaning the n-th power of the prime p.

These definitions are very basic and informal. The reader can think of a field as a set with some "nice properties", that will make it easier to work with. We will focus our attention in a more specific kind of field.

Definition 9.8. An *algebraically closed field* (or ACF) is a field F in which any non-constant polynomial with all coefficients in F has at least one root in F.

Definition 9.9. The *characteristic* of an algebraically closed field is the least number p, such that $x + \overset{p \text{ times}}{\ldots} + x = 0$ for all x in such field (it is actually a bit more general than this, but this is the case we are interested in). If such p does not exist, we say the field has characteristic 0.

Definition 9.10. The *transcendence degree* of an ACF is the cardinality of the set of *transcendental* elements (i.e. elements that are not a root of any polynomial).

Definition 9.11. When we refer to ACF of a certain characteristic p, we will denote this as ACF_p.

In this paper, the theory of ACF will be of our interest. To obtain the theory of ACF_p, we will consider the sentence ψ_p for p a prime number. Let ψ_p be:

$$\forall x (x \overset{p \text{ times}}{+ \ldots +} x) = 0.$$

Since neither ψ_p or $\neg\psi_p$ are implied by ACF, we add them as axioms. Let ACF_p be ACF plus ψ_p (ACF_0 will be ACF plus $\neg\psi_p$ for every prime p).

9.4. **Zorn's Lemma.**

Definition 9.12. If X is a set and $<$ is a binary relation on X, we say that $(X,<)$ is a *partial order* if $(X,<) \models \forall x \neg(x < x)$ and $(X,<) \models \forall x \forall y \forall z((x < y \wedge y < z) \longrightarrow x < z)$.

Definition 9.13. We say that $(X,<)$ is a *linear order* if we have a partial order and $(X,<) \models \forall x \forall y(x < y \vee x = y \vee y < x)$.

Definition 9.14. If $(X,<)$ is a partial order, then we say that $C \subseteq X$ is a *chain* in X if C is linearly ordered by $<$.

Theorem 9.15 (Zorn's Lemma)**.** *If $(X,<)$ is a partial order such that for every chain $C \subseteq X$ there is $x \in X$ such that $c \leq x$ for all $c \in C$, then there is $y \in X$ such that there is no $z \in X$ with $z > x$. In other words, if every chain has an upper bound, then there is a maximal element of X.*

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