Doob's Martingale Convergence Theorems

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Background

1. Discovered and named after the American mathematician, Joseph L. Doob
2. The martingale convergence theorems are results on the limits of martingales: more specifically, the theorems state that **all supermartingales satisfying a certain type of bound must converge** (will define these terms in the next slide).
3. Very useful result in probability theory that has applications in game theory, gambling, and the stock market.

Definitions

- **Random Variable:** a variable whose value is dependent on random events.
  - A random variable X is a function that maps the outcome of a random event in a sample space Ω, to a measurable space E, often the real numbers
  - $X{:}\Omega \longrightarrow E$
  - Discrete vs continuous

Example

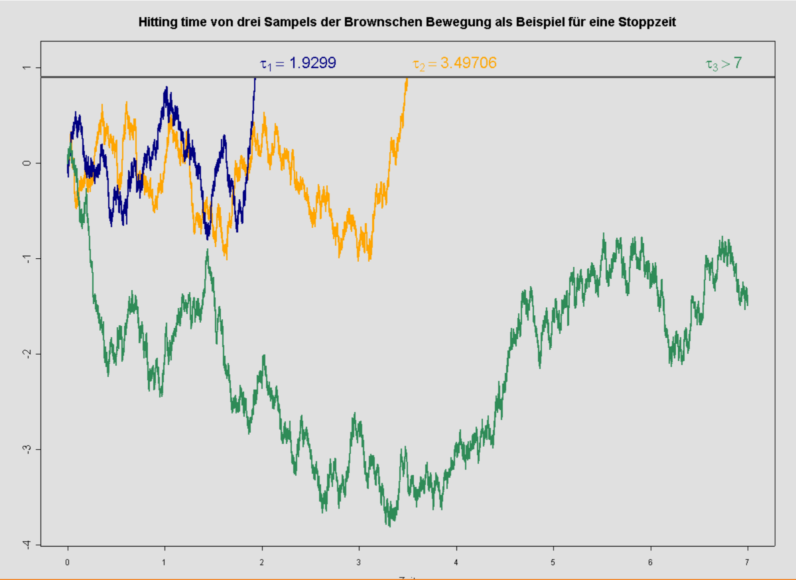
When rolling a dice, the outcome of the roll is a random variable because it's mapping the result of a random event (rolling a dice) in a sample space where each outcome has a $\frac{1}{6}$ chance, to an integer between 1 and 6

Definitions

Martingale: a sequence of random variables $X_1, X_2, \cdots X_n$ such that, at any particular time, the expected value of the next random variable is equal to the present value, regardless of prior values.

- $E(X_{n+1}|X_1, X_2, \cdots, X_n) = X_n$
- Submartingales: $E(X_{n+1}|X_1, X_2, \cdots, X_n) \geq X_n$
- Supermartingales: $E(X_{n+1}|X_1, X_2, \cdots, X_n) \leq X_n$

Visualizing Martingales



Examples of Martingales

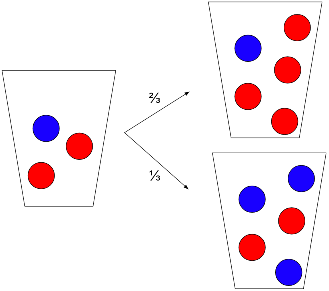
- An unbiased $\pm n$ random walk in any number of dimensions
  - If the current value is c, then the next value is either $c + n$ or $c - n$ with equal probability, therefore making the expected value c.

- A gambler's fortune, given the game is fair.
  - The fortune X_n after n rounds of a fair game with a bet of m dollars is $X_{n-1} \pm m \Rightarrow E(X_n) = \frac{(X_{n-1}+m)}{2} + \frac{(X_{n-1}+m)}{2} = X_{n-1}$ satisfying the conditions of a martingale.

Examples of Martingales

Polyas Urn

- Different colored marbles in an urn such that at each iteration a random marble is chosen and replaced with more marbles of that color
- The fraction of marbles of a given color in the urn is a martingale.



Example

The fraction of blue marbles at the start is $\frac{1}{3}$ and the expected fraction of blue marbles after the iteration is $(\frac{2}{3})(\frac{1}{5}) + (\frac{1}{3})(\frac{3}{5}) = \frac{1}{3}$, which makes the fraction of blue marbles a martingale.

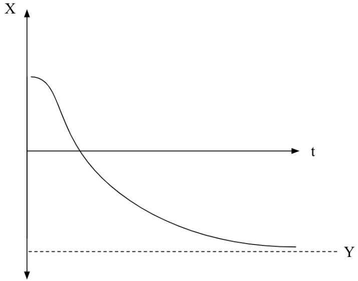
Doob's First Martingale Convergence Theorem

Theorem

Let $X_1, X_2, X_3 \cdots$ *be a supermartingale such that*

$$\sup_{t \in \mathbf{N}} \mathrm{E}[X_t^-] < \infty$$

where $X_t^- = -\min(X_t, 0)$, *then the sequence will converge to a random variable* Y*.*



Note

There are symmetric results for submartingales

Non-Rigorous Proof Sketch

There are two reasons why a sequence may fail to converge:

- It goes off to infinity
- It oscillates

- The first is impossible due to the $\sup_{t \in \mathbf{N}} \mathrm{E}[X_t^-] < \infty$ bound.
- The second can be proven impossible through contradiction.
  - Model a supermartingale as stock market game where the expected change in stock price is at most zero.
  - No strategy can return positive expected profit
  - If prices oscillated without converging, buying low and selling high would return a positive expected profit, hence a **contradiction**

Doob's Second Martingale Convergence Theorem

Theorem

Let $N_1, N_2, N_3 \cdots$ *be a supermartingale that is uniformly integrable, then there exists an integrable random variable* N *such that*

$$\lim_{t \to \infty} N_t = N$$

and

$$E(|N_t - N|) \to 0$$

Related theorems

Monotone Convergence Theorem

If a sequence is monotonically increasing and bounded by a supremum, it will converge to said supremum. By symmetry, the same applies to monotonically decreasing sequence that is bounded by an infimum.

Optional Stopping Theorem

Under certain conditions, the expected value of a martingale at a stopping time is equal to its initial expected value.

Thank you for listening!

Please message me in discord with any questions.