Analyzing Results of Doobs Martingale Convergence Theorems

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Abstract

In this paper, I will explain and provide all the definitions and background necessary to fully understand the results that arise from Doobs Martingale Convergence Theorems. I will then go into detail about all aspects of the theorems and also prove them. Lastly, I will reflect on many of the applications of theorems and their usefulness to the mathematical community.

1 Introduction

The first evidence of humans gambling can be seen as far back as the Paleolithic period, hundreds of thousands of years ago. Ever since then, different strategies and concepts have arisen to attempt to increase ones odds of making money. Eventually, it was realized that mathematics had a large presence in gambling and that it could be used to attempt to predict and analyze the outcomes of bets and possibly give gambler's that understood the mathematics behind a game an advantage over a player that was oblivious to such math. This motivated the development of many math-based betting strategies, a popular class of these strategies being martingale betting strategies.

Originated from and popular in 18th century France, the term **martingale** originally referred to a class of betting strategies for fair games where the gambler has a $\frac{1}{2}$ chance of winning their stake and $\frac{1}{2}$ chance of losing their stake. The martingale betting strategies involve the gambler doubling his stake each round so the first win would recover all previous losses while also winning a profit equal to the original stake. As time and the gamblers expected wealth jointly approach infinity, the probability of eventually winning the stake approaches 1, which makes the martingale betting strategy appear to practically guarantee an eventual win. However, the exponential growth of the bets will eventually bankrupt the gambler due to finite budget. Observe that after each iteration of the game, intuition tells us that the gamblers expected wealth should be equal to the gamblers wealth prior to the iteration because it's equally likely to increase by some value as it is to decrease by that value which leads to an expected change of 0. This facet is what gives the betting strategy the name martingale.

Introduced to probability theory in 1934 by Paul Lévy, martingales were defined as a sequence of **random variables** such that the expected value of the next random variable is equal to the previous one. In probability theory, the definition of martingales were expanded to **submartingales** and **supermartingales** where submartingales are defined as sequences of random variables such that the next value is expected to be larger than or equal to the prior one and supermartingales being a sequence of random variables such that the next value is expected to be less than or equal to the prior one. Most of the original development on the theory of martingales was done by the American mathematician Joseph Leo Doob.

In this paper, I will be focusing on Doob's theorems regarding martingale convergence which are a collection of results on the limits of supermartingales. Roughly speaking, the theorems state that if we have a supermartingale which is bounded by some finite supremum, it almost surely converges to some random variable[2]. However, just because the martingale converges to a random variable, doesn't mean the expected values of the random variables in the martingale will approach the expected value of the random variable the martingale is converging to. In order for this to occur, Doobs second convergence theorem states that the condition of **uniform integrability** of the random variables must hold. I will formally state the theorems later in my paper and define all terms necessary to understand it. I will also explore the applications of the martingale theorems in fields such as game theory, finance, and gambling and how the theorems can be used to produce well-known results such as **Lévy's zero–one law**. Lastly, I'll reflect on the significance of Doobs theorems and their importance to the mathematical community.

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3 Preliminaries

Definition 3.1 (Random Variable)**.** A random variable X is a function that maps the outcome of a random event in a sample space Ω, to a measurable space E, often the real numbers ($X{:}\Omega \longrightarrow E$). Random variables are often denoted as a capital roman letters such as X, Y, Z, T.

Definition 3.2 (Martingale)**.** A martingale is a sequence of random variables $X_1, X_2, \cdots X_n$ such that

$$E(X_{n+1}|X_1, X_2, \cdots, X_n) = X_n$$

At any particular time, the expected value of the next random variable is equal to the present value, regardless of prior values.

Definition 3.3 (Submartingale)**.** A submartingale is a sequence of random variables $X_1, X_2, \cdots X_n$ such that

$$E(X_{n+1}|X_1, X_2, \cdots, X_n) \geq X_n$$

At any particular time, the expected value of the next random variable is greater than or equal to the present value, regardless of prior values. Analogous to a monotonically increasing sequence.

Definition 3.4 (Supermartingale)**.** A supermartingale is a sequence of random variables $X_1, X_2, \cdots X_n$ such that

$$E(X_{n+1}|X_1, X_2, \cdots, X_n) \leq X_n$$

At any particular time, the expected value of the next random variable is less than or equal to the present value, regardless of prior values. Analogous to a monotonically decreasing sequence.

Definition 3.5 (Supremum)**.** The supremum (abbreviated sup) of a subset S of a partially ordered set P is the smallest element in P that's greater than or equal to each element in S, if the element exists

Definition 3.6 (Infimum)**.** the infimum (abbreviated inf) of a subset S of a partially ordered set P is a greatest element in P that is less than or equal to each element of S, if such an element exists.

Definition 3.7 (Uniform Integrability)**.** The random variables $X_1, X_2, X_3 \cdots X_n$ in a martingale are said to be uniformly integrable if there exists $K \in [0, \infty)$ such that $E(|X| I_{|X| \geq K}) \leq \varepsilon$ for all X in the martingale and where $I_{|X| \geq K}$ is the indicator function:

$$I_{|X| \geq K} = \begin{cases} 1 & \text{if } |X| \geq K, \\ 0 & \text{if } |X| < K. \end{cases}.$$

4 Doob's First Martingale Convergence Theorem

The first of Doob's convergence theorems states that

Theorem 1 (Doob's First Martingale Convergence Theorem)**.** *Let $X_1, X_2, X_3, \ldots$ be a supermartingale. Suppose that the supermartingale has the bound*

$$\sup_{t \in \mathbf{N}} \mathrm{E}[X_t^-] < \infty$$

where $X_t^- = -\min(X_t, 0)$, then the sequence will almost surely converge to a random variable X.

Observe that through symmetry, there are identical results for submartingales with bounded expectation of the positive part.

The following is a sketch proof of the theorem.

Proof. We can prove the following result by contradiction. For a sequence to not converge it either goes off to infinity or it oscillates. The first is impossible due to the $\sup_{t \in \mathbf{N}} \mathrm{E}[X_t^-] < \infty$ bound. To understand why the second is impossible we can model a supermartingale as stock market game where the expected change in stock price is at most zero. There is no strategy that can return w positive expected profit since the expected change must be less than or equal to 0. If prices oscillated without converging, buying low and selling high would return a positive expected profit, hence a **contradiction**.[1] □

5 Failure of convergence

Note that, while Doob's first martingale convergence theorem guarantees the martingale converging to some random variable, it doesn't guarentee that the expected values of the random variables in the martingale converge to the expected value of the random variable the martingale is converging to. In other words we can't guarantee:

$$E(X_1), E(X_2), E(X_3) \cdots \to E(X)$$

which is more formally known as converging in mean. It can also be expressed as

$$\lim_{n\to\infty} E[|X_n - X|] = 0$$

In order to ensure convergence in mean, uniform integrability is necessary which brings us to Doobs second martingale convergence theorem.

6 Doob's Second Martingale Convergence Theorem

The statement of Doob's second convergence theorem states that

Theorem 2. *Let $N_1, N_2, N_3 \cdots$ be a supermartingale that is uniformly integrable, then there exists an integrable random variable N such that*

$$\lim_{t\to\infty} N_t = N$$

and

$$E(|N_t - N|) \to 0$$

Observe that through symmetry, there are identical results for supermartingales.

Note that in Doob's first martingale convergence theorem the convergence is pointwise while in Doob's second martingale convergence theorem there is convergence in mean.

This result is a lot more difficult to prove than the first martingale theorem.

7 Applications

Doob's martingale convergence theorems are very useful results in probability theory that have many applications in game theory, gambling, finance, and the stock market. Yet, the convergence theorems are also useful in solving math problems involving martingales and many famous results can be derived from them. An example is **Levy's Zero-One law**, which claims that if the information that determines the outcome of an event is gradually being learned, then the outcome of the event will gradually become certain.

References

[1] CHARLES W. LAMB. A short proof of the martingale convergence theorem, 1973.

[2] Wikipedia contributors. Doob's martingale convergence theorems — Wikipedia, the free encyclopedia, 2021.