The Bombieri-Vinogradov Theorem

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Euler Circle

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Outline

1. The Mystery of Primes.
2. Error term in the PNT.
3. The Bombieri-Vinogradov Theorem.

The atoms of natural numbers

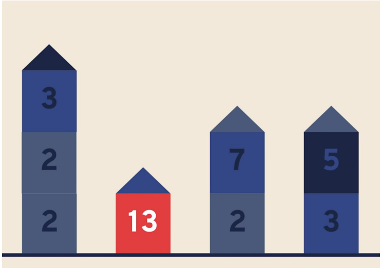

Figure: "Factor City, Quanta Magazine - Riemann Hypothesis"

A *prime* is a natural number greater than 1 that is divisible only by 1 and itself.

Primes are "the" building blocks of natural numbers.

Pattern(s) in primes?

Primes are particularly known for their infamous random distribution and have perplexed mathematicians for centuries.

Mersenne's conjecture, 1644

$2^p - 1$ is a prime where p is also a prime.

Fermat's conjecture, 1650

$2^{2^n} + 1$ is a prime for all positive integers.

Distribution of Primes

While the "Mystery of Primes" has been unsolved till date, a significant amount of progress has been made since the Greeks.

How many primes exist less than a positive integer x?

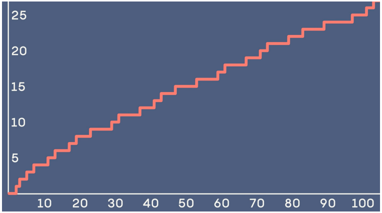

Figure: "Prime-counting step function, Carl Friedrich Gauss"

Gauss's conjecture(1792)

Primes less than a positive integer x are roughly equal to $\frac{x}{\log x}$.

Primes $\leq x$

Prime Number Theorem (Hadamard and de la Valée Poussin, 1896)

$$\pi(x) \sim \int_2^x \frac{dx}{\log x} \sim \frac{x}{\log x}.$$

Prime Number Theorem for primes in APs. (Dirichlet, Legendre)

$$\pi(x; q, a) \sim \frac{\pi(x)}{\phi(q)} \sim \frac{1}{\phi(q)} \frac{x}{\log x},$$

where,

$$\pi(x; q, a) = \sum_{\substack{p \leq x \\ p \equiv a \bmod q \\ (a,q)=1}} 1.$$

Twin Primes

A *twin prime* is a pair of two consecutive primes whose difference is equal to 2. Examples of twin primes include (3,5), (5,7), and (11,13).

Polignac's Conjecture (de Polignac, 1849)

There are infinitely many pairs of consecutive primes of the form $(p, p + h)$, where h is a positive even number.

This is also famously known as the **Twin Prime conjecture** when $h = 2$.

Gaps between Primes

Theorem (Goldston, Pintz, and Yildrim, 2005)

There are infinitely many pairs of primes that differ by at most 16 **if** the Elliot-Halberstam conjecture is true.

Theorem (Zhang, 2013)

There are infinitely many pairs of primes that are at most 70,000,000 apart.

PNT revisited

Analogous to the prime-counting function for primes in arithmetic progressions, $\theta(x; q, a)$ is defined as

$$\theta(x; q, a) = \sum_{\substack{p \leq x \\ p \equiv a \bmod q}} \log p.$$

Identical to π and θ, $\psi(x; q, a)$ is defined as

$$\psi(x; q, a) = \sum_{\substack{p^k \leq x \\ p^k \equiv a \bmod q}} \log p.$$

Note: π can be replaced by θ, which in turn can be replaced by ψ.

Precise enough?

Asymptotic Inequality for π is not *precise* enough.

$$\pi(x; q, a) = \frac{1}{\phi(q)} \frac{x}{\log x} + o\left(\frac{x}{\log x}\right)$$

The goal is to replace the error term with a more precise *big-oh* estimate.

Siegel-Walfiz Theorem

Gives a big-oh estimate of $O(xe^{-C(A)\sqrt{\log x}})$ for the error term in PNT for primes in arithmetic progressions.

Assuming *GRH*, we get a big-oh estimate $O(x^{1/2})$ ignoring logarithmic factors.

The Bombieri-Vinogradov Theorem.

Let $E(x;q)$ represent the maximum possible error in the PNT for any congruence class modulo q for numbers $< x$.

The Bombieri-Vinogradov Theorem

$$\sum_{q \leq Q} E(x;q) \ll x\left(\log xQ\right)^2 + Q\log x, \tag{0.1}$$

where, $Q \leq \sqrt{x}$ for positive real numbers x and Q.

Gaps between primes revisited

Elliot-Halberstam conjecture is actually a generalisation of the Bombieri-Vinogradov Theorem!

For every $\theta < 1$ and $A > 0$ there exists a constant $C > 0$ such that

$$\sum_{1 \leq q \leq x^{\theta}} E(x; q) \leq \frac{Cx}{A \log x},$$

for all $x > 2$.

Assuming the Elliot-Halberstam conjecture is proved to be true,

1. There are infinitely many pairs of primes that differ by at most 16 according to Goldston, Pintz, and Yildrim Theorem.
2. (Maynard, 2013) There are infinitely many primes that differ by at most 12.
3. (Polymath, 2014) There are infinitely many pairs of consecutive primes that differ by at most 6.

Remarks

1. Dirichlet Characters
2. Pólya-Vinogradov Inequality.
3. Sieve Theory and the Large Sieve Inequality.
4. Barban-Davenport-Halberstam Theorem.

Thank you for listening!!