Exploring Ehrhart Theory

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What is Ehrhart Theory?

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- Counting lattice points in polytopes (higher-dimensional generalization of polygons and polyhedra)
- Relating discrete volume and continuous volume

Pick's Theorem

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One of the most famous theorems involving lattice-point enumeration is *Pick's theorem*, published by Austrian mathematician Georg Alexander Pick in 1899.

Theorem (Pick's theorem)

Given a convex integral polygon $\mathcal{P}$, let the number of lattice points strictly interior to $\mathcal{P}$ be I, and let the number of lattice points on the boundary of $\mathcal{P}$ be B. Then, the formula

$$A = I + \frac{B}{2} - 1$$

gives the area A of $\mathcal{P}$.

Beyond Pick

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Ehrhart developed Ehrhart theory as a way to generalize Pick's theorem using an alternative formulation.

Main Concepts

The Central Question

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Question

What is the number of lattice points contained in $t\mathcal{P}$ in terms of t and the properties of $\mathcal{P}$?

Lattice-Point Enumerators

To answer this question, we define the *lattice-point enumerator* function of $\mathcal{P}$, denoted by $L_{\mathcal{P}}(t)$.

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Definition

The lattice-point enumerator of $\mathcal{P} \subset \mathbb{R}^d$, which counts the number of lattice points inside $t\mathcal{P}$ when evaluated at t, is

$$L_{\mathcal{P}}(t) = \left| t\mathcal{P} \cap \mathbb{Z}^d \right|.$$

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The value of $L_{\mathcal{P}}(t)$ is also called the *discrete volume* of $t\mathcal{P}$.

Ehrhart Series of Polytopes

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Definition

The Ehrhart series of a polytope $\mathcal{P}$, denoted $\mathrm{Ehr}_{\mathcal{P}}(z)$, is the generating function of $L_{\mathcal{P}}(t)$ where

$$\mathrm{Ehr}_{\mathcal{P}}(z) := \sum_{t \geq 0} L_{\mathcal{P}}(t) z^t.$$

Ehrhart's Theorem

The following is the most important fact about lattice-point enumerators:

Theorem (Ehrhart's theorem)

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Because of this result, lattice-point enumerators are often called *Ehrhart polynomials*.

Return to Pick's — Ehrhart Polynomials of Polygons

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Theorem

Given a convex integral polygon $\mathcal{P}$, let its area be A and let the number of lattice points on its boundary be B. Then,

$$L_{\mathcal{P}}(t) = At^2 + \frac{B}{2}t + 1.$$

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$$L_{\mathcal{P}}(t) = At^2 + \frac{B}{2}t + 1.$$

This result follows easily from Pick's Theorem after proving that the area of $t\mathcal{P}$ is At^2 and the number of lattice points on the boundary of $t\mathcal{P}$ is Bt.

An Example

The Unit d-Hypercube

One common type of polytope is the *hypercube*. A hypercube in higher dimensions is the generalization of the 2-dimensional square and the 3-dimensional cube. A *unit d-hypercube* is a d-hypercube with side length 1.

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Definition

We formally define the unit d-hypercube, denoted by $\square_d$, as the polytope whose vertices are all of the points in $\mathbb{R}^d$ such that every coordinate is either 0 or 1:

$$\begin{aligned}\square_d &:= \mathrm{conv}\{(x_1, x_2, \ldots, x_d) \in \mathbb{R}^d : x_i = 0 \text{ or } 1 \text{ for } 1 \leq i \leq d\} \\ &= \{(x_1, x_2, \ldots, x_d) \in \mathbb{R}^d : 0 \leq x_i \leq 1 \text{ for } 1 \leq i \leq d\}.\end{aligned}$$

The Unit d-Hypercube

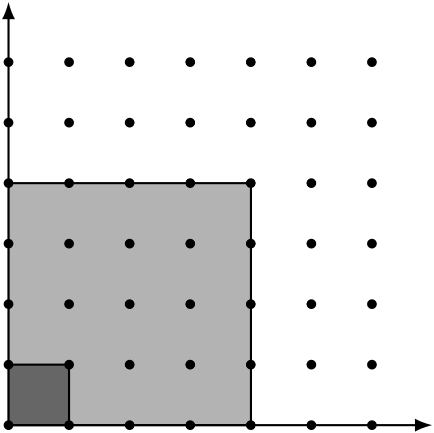

Figure: $\square_2$ and $4\,\square_2$ on the $\mathbb{Z}^2$ lattice.

Ehrhart Polynomial of the Unit d-Hypercube

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Proof.

In $t\square_d$, each coordinate of a lattice point must be an integer in the interval $[0, t]$. There are $t+1$ such integers and d coordinates, so there are $(t+1)^d$ lattice points. ■

Discrete and Continuous Volume

Volume in Higher Dimensions

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As s approaches 0, the hypercubes approach points and the approximation approaches the actual (continuous) volume.

The Leading Coefficient

What is the actual relationship between a polytope's discrete and continuous volumes? The answer lies in the Ehrhart polynomial of the polytope.

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Theorem

For a given convex integral polytope $\mathcal{P} \subset \mathbb{R}^d$, let its Ehrhart polynomial be

$$L_{\mathcal{P}}(t) = a_d t^d + a_{d-1} t^{d-1} + \cdots + a_1 t + 1.$$

Then, a_d equals the volume of $\mathcal{P}$.

Open Problems

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3. There does not exist a standard algorithm for determining either of these properties — proving these properties often uses very different methods for different polytopes.
4. Of course there are many other open problems

Thank you!