EXPLORING EHRHART THEORY

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Abstract. In this paper, we will overview the fundamentals of Ehrhart theory, which studies the number of lattice points contained by a polytope. Ehrhart theory is named after Eugène Ehrhart, who showed in 1962 that number of lattice points contained by the t^{th} dilate of a d-dimensional polytope is a rational polynomial in t of degree d, called the Ehrhart polynomial. We will examine the Ehrhart polynomials of some common polytopes and the proof of Ehrhart's theorem, as well as properties of Ehrhart polynomials and open problems in the field.

1. Introduction

Oftentimes in mathematics, there is a clear distinction in the way we approach a field of study: either continuously or discretely. Branches of mathematics like combinatorics and number theory are generally studied as parts of discrete math whereas algebra and geometry are usually continuous. However, there exists certain fascinating subjects in math that connect the worlds of the discrete and the continuous. *Ehrhart theory* is one such subject.

Ehrhart theory, named after French mathematician Eugène Ehrhart, studies the relationship between an object's *continuous volume* — the "normal" or "intuitive" sense of volume — and its *discrete volume* — a different sense of volume determined by the number of *lattice points* inside the object, where a lattice point is a point whose coordinates are integers.

The history of Ehrhart theory very much has to do with lattice points. In the mathematics of lattice points, perhaps the most well-known result is *Pick's theorem* (Theorem 5.1), which gives the area of a convex polygon with lattice-point vertices in terms of the number of lattice points inside it and on its boundary. However, Pick's theorem fails to generalize into higher dimensions. For this purpose, Ehrhart devised a different approach.

Ehrhart instead studied how the number of lattice points inside an object changed as the object was scaled up in size. For this purpose, he defined the *lattice-point enumerator* function in t of an object, which counted the number of lattice points in the object after being scaled up by a factor of t for positive integers t. He discovered the central theorem of Ehrhart theory — that the lattice-point enumerator of convex *polytopes* (polygons and polyhedra generalized to higher dimensions) with interger vertices is a rational polynomial in t whose degree equals the dimensionality of the polytope. Today, this result is called *Ehrhart's theorem* (Theorem 6.1).

Connecting back to Pick's theorem and volume, Ehrhart discovered that the leading coefficient of a polytope's lattice-point enumerator always equaled the polytope's volume. In a sense, this served as a generalization of Pick's theorem into higher dimensions. This relationship between continuous volume and discrete lattice points connects continuous and discrete mathematics and brings together many different branches of mathematics, including combinatorics, geometry, and number theory.

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Ehrhart's work was and still is being expanded upon by other mathematicians, eventually developing into its own field of study, which we today call Ehrhart theory. One of the most notable contributions to Ehrhart theory came from British mathematician Ian G. Macdonald, who proved a relationship between the lattice-point enumerators of a polytope and its interior, a theorem called *Ehrhart-Macdonald Reciprocity* (Theorem 7.2).

The aim of this expository paper is to explore the fundamentals of Ehrhart theory and provide the reader an introduction to the basic ideas and main results. To do this, we will follow the following structure:

We will begin in Section 2 by formally introducing *polytopes* and relevant terminology so as to be able to discuss higher-dimensional objects. Then, in Section 3, we will more precisely define the *lattice-point enumerator* and compute the lattice-point enumerators of several common polytopes. Continuing in Section 4, we will explore the role that *generating functions* play in Ehrhart theory, and define the *Ehrhart series* of a polytope. Next, we will have an in-depth look at *Pick's theorem*, provide a traditional proof of it, and further explore its connections to Ehrhart theory in Section 5. In Section 6, we will get ready to discuss the main results of Ehrhart theory, including *Ehrhart's theorem* and *Ehrhart-Macdonald Reciprocity*, by introducing and proving several lemmas that will be needed for our proof of Ehrhart's theorem. Then, in Section 7, we will derive the proof of Ehrhart's theorem using our preparation earlier, as well as formally state Ehrhart-Macdonald Reciprocity. Lastly, in Section 8, we will examine *Ehrhart positivity*, an open area of research in Ehrhart theory.

2. Polytopes

We begin with the formal introduction to *polytopes*. Since Ehrhart theory is not restricted to just shapes with low dimensionalities, we need to first define the mathematics of polytopes, the higher-dimensional generalization of the 2-dimensional polygon or the 3-dimensional polyhedron. In this paper, we will only be concerned with convex polytopes. In general, there are two different yet equivalent ways to define convex polytopes: the *vertex description* and the *hyperplane description*.

Using the vertex description, a convex polytope $\mathcal{P} \subset \mathbb{R}^d$ is the *convex hull* of a finite set of points $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ in $\mathbb{R}^d$. We denote this by $\mathcal{P} = \operatorname{conv}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$.

Definition 2.1. For a finite set of points $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ in $\mathbb{R}^d$, its convex hull is

$$\operatorname{conv}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\} \\ \coloneqq \{\lambda_1\mathbf{v}_1 + \lambda_2\mathbf{v}_2 + \cdots + \lambda_n\mathbf{v}_n : \lambda_i \geq 0 \text{ for } 1 \leq i \leq n \text{ and } \lambda_1 + \lambda_2 + \cdots + \lambda_n = 1\}.$$

If we instead use the hyperplane description, a convex polytope $\mathcal{P} \subset \mathbb{R}^d$ is the bounded intersection of finitely many d-dimensional *half-spaces* and $(d-1)$-dimensional *hyperplanes*.

Definition 2.2. A hyperplane $H \subset \mathbb{R}^d$ is a $(d-1)$-dimensional subspace of a d-dimensional space. A half-space $\mathcal{H} \subset \mathbb{R}^d$ is the part of a d-dimensional space that lies on a given side of a $(d-1)$-dimensional hyperplane. Formally,

$$H \coloneqq \{\mathbf{x} \in \mathbb{R}^d : \mathbf{a} \cdot \mathbf{x} = b\}$$

for some $\mathbf{a} \in \mathbb{R}^d$ and some constant b, and

$$\mathcal{H} \coloneqq \{\mathbf{x} \in \mathbb{R}^d : \mathbf{a} \cdot \mathbf{x} \geq b\} \text{ or } \{\mathbf{x} \in \mathbb{R}^d : \mathbf{a} \cdot \mathbf{x} \leq b\}$$

for some $\mathbf{a} \in \mathbb{R}^d$ and some constant b.

We call a hyperplane H a *supporting hyperplane* of a polytope $\mathcal{P}$ if $\mathcal{P}$ is completely contained in one of the two half-spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ bounded by H, that is, $\mathcal{P} \subset \mathcal{H}_1$ or $\mathcal{P} \subset \mathcal{H}_2$.

In addition, we can also define the surface regions of a polytope analogous to the surface regions of a polyhedron. A *face* of a polytope $\mathcal{P}$ is the set of points $\mathcal{P} \cap H$, where H is a supporting hyperplane of $\mathcal{P}$. Note that a face of a d-dimensional polytope can have any dimensionality less than or equal to d. In particular, the $(d-1)$-dimensional faces are called *facets*, the 1-dimensional (line segment) faces are called *edges*, and the 0-dimensional (point) faces are called *vertices*.

Ehrhart theory is concerned with polytopes with integer or rational vertices. As such, we define an *integral polytope* as a polytope whose vertices all have integer coordinates. Similarly, if the vertices all have rational coordinates, then the polytope is a *rational polytope*.

Finally, Ehrhart theory also studies what happens when a polytope is scaled up in size. A polytope $\mathcal{P}$ scaled up by a factor of t is called its t^{th} dilate, denoted by $t\mathcal{P}$.

Definition 2.3. For a positive integer t, the t^{th} dilate of a polytope $\mathcal{P} \subset \mathbb{R}^d$ is $t\mathcal{P}$, and

$$\begin{aligned} t\mathcal{P} &= \{(tx_1, tx_2, \ldots, tx_d) : (x_1, x_2, \ldots, x_d) \in \mathcal{P}\} \\ &= \{t\mathbf{x} : \mathbf{x} \in \mathcal{P}\}. \end{aligned}$$

With the fundamental language of polytopes, we can now explore the core of Ehrhart theory.

3. Lattice-Point Enumeration

3.1. **Lattice-Point Enumerators.** The central theme of Ehrhart theory is counting the number of *lattice points* — points with only integer coordinates — contained within a polytope $\mathcal{P}$. Specifically, we are interested in how the number of lattice points inside a polytope changes as it is scaled up:

Question 3.1. *What is the number of lattice points contained in $t\mathcal{P}$ in terms of t?*

To answer this question, we define the *lattice-point enumerator* function, also called the *Ehrhart polynomial* for reasons we will discuss later, of $\mathcal{P}$, denoted by $L_{\mathcal{P}}(t)$.

Definition 3.2. The lattice-point enumerator of $\mathcal{P} \subset \mathbb{R}^d$, which counts the number of lattice points inside $t\mathcal{P}$ when evaluated at t, is

$$L_{\mathcal{P}}(t) = \left| t\mathcal{P} \cap \mathbb{Z}^d \right|.$$

The value of $L_{\mathcal{P}}(t)$ is also called the *discrete volume* of $t\mathcal{P}$.

To better understand this function, we can examine a few examples.

3.2. **The Unit *d*-Cube.** One of the polytopes with the simplest lattice-point enumerators is the *unit d-cube*, the generalization of the 2-dimensional unit square and the 3-dimensional unit cube.

Definition 3.3. The unit d-cube, denoted by $\square_d$, is the polytope whose vertices are all of the points in $\mathbb{R}^d$ such that every coordinate is either 0 or 1:

$$\begin{aligned} \square_d &:= \operatorname{conv}\{(x_1, x_2, \ldots, x_d) \in \mathbb{R}^d : x_i = 0 \text{ or } 1 \text{ for } 1 \le i \le d\} \\ &= \{(x_1, x_2, \ldots, x_d) \in \mathbb{R}^d : 0 \le x_i \le 1 \text{ for } 1 \le i \le d\}. \end{aligned}$$

We can also consider strictly the *interior* of a polytope, denoted by $\mathcal{P}^\circ$ for a polytope $\mathcal{P}$. The interior of the unit d-cube is

$$\square_d^\circ = \{(x_1, x_2, \ldots, x_d) \in \mathbb{R}^d : 0 < x_i < 1 \text{ for } 1 \le i \le d\}.$$

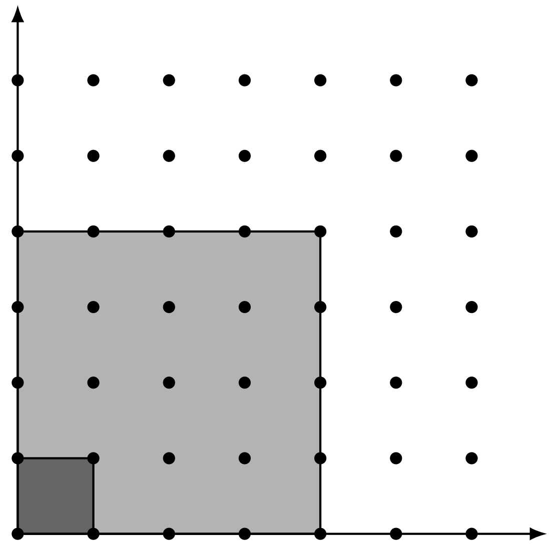

Figure 1. $\square_2$ and $4\,\square_2$ on the $\mathbb{Z}^2$ lattice.

Figure 1 demonstrates that in the case of $d = 2$, it is apparent that $L_{\square_2}(t) = (t+1)^2$. In general, the lattice-point enumerators of the unit d-cube and its interior are given by the following theorem.

Theorem 3.4. *The lattice-point enumerator of the unit d-cube is*

$$L_{\square_d}(t) = (t+1)^d,$$

and the lattice-point enumerator of the interior of the unit d-cube is

$$L_{\square_d^\circ}(t) = (t-1)^d.$$

Proof. A given lattice point $(x_1, x_2, \ldots, x_d)$ in $t\,\square_d$ can have $x_i = 0, 1, \ldots, t$ for each $1 \le i \le d$. Since each of the d coordinates has $t+1$ possible values, the lattice-point enumerator of the unit d-cube is $L_{\square_d}(t) = (t+1)^d$.

For the interior of the d-cube, each of the coordinates of a lattice point in $t\,\square_d^\circ$ instead only has $t-1$ possible values, so the lattice-point enumerator of the interior is $L_{\square_d^\circ}(t) = (t-1)^d$. ∎

3.3. **The Standard *d*-Simplex.** Another common type of polytope is the *simplex*. A simplex in higher dimensions is the generalization of the 2-dimensional triangle and the 3-dimensional tetrahedron. The name "simplex" refers to the fact that the d-simplex is the "simplest" polytope in d-dimensions, as the d-simplex has $d+1$ vertices and $d+1$ facets, the minimum possible amounts.

In particular, we are interested in the *standard d-simplex*, defined as the convex hull of the origin and the d unit vectors. Figure 2 shows dilates of the standard 2-simplex, which is simply a right isosceles triangle.

Definition 3.5. The standard d-simplex, denoted by Δ_d, is the polytope whose vertices are the origin and the d unit vectors $\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_d$ (specifically, $\mathbf{e}_i$ is the point where the i^{th} coordinate is 1 and all other coordinates are 0):

$$\begin{aligned}\Delta_d &\coloneqq \operatorname{conv}\{\mathbf{0}, \mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_d\}\\ &= \{(x_1, x_2, \dots, x_d) \in \mathbb{R}^d : x_1 + x_2 + \cdots + x_d \le 1 \text{ and } x_i \ge 0 \text{ for } 1 \le i \le d\}.\end{aligned}$$

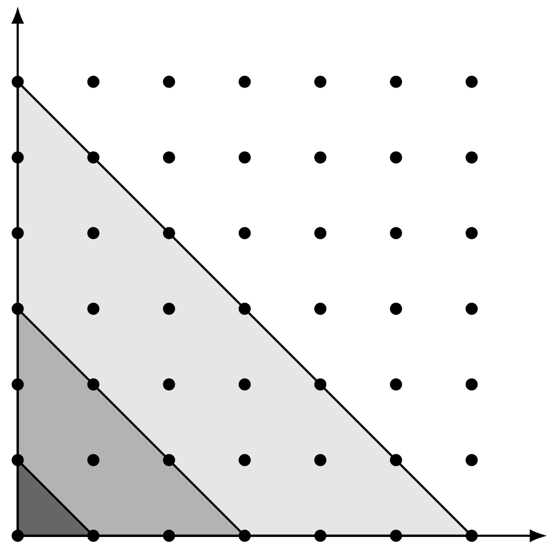

Figure 2. Δ_2, $3\Delta_2$, and $6\Delta_2$ on the $\mathbb{Z}^2$ lattice.

Theorem 3.6 gives the lattice-point enumerator of Δ_d.

Theorem 3.6. *The lattice-point enumerator of the standard d-simplex is*

$$L_{\Delta_d}(t) = \binom{d+t}{d}.$$

Proof. A lattice point $(x_1, x_2, \dots, x_d)$ in $t\Delta_d$ satisfies

$$x_1 + x_2 + \cdots + x_d \le t \tag{3.1}$$

where x_i is a nonnegative integer for all i. We can transform (3.1) into an equality by introducing a slack variable $x_{d+1} \in \mathbb{Z}_{\ge 0}$, which represents the difference between the left hand side and the right hand side of (3.1). This gives

$$x_1 + x_2 + \cdots + x_d + x_{d+1} = t. \tag{3.2}$$

Counting the number of solutions to (3.2) is equivalent to the classical combinatorial problem of counting the number of ways to place n indistinguishable objects into k distinguishable bins, and in this case we have $n = t$ and $k = d + 1$. By the well-known Stars and Bars (also called Sticks and Stones, etc.) method, there are

$$\binom{n+k-1}{k-1}$$

ways to place the objects.

Making the substitution $n = t$ and $k = d + 1$ gives that $\binom{d+t}{d}$ is the number of solutions to (3.2) and (3.1) as well as the number of lattice points in $t\Delta_d$. ■

3.4. **The d-Cross-Polytope.** The last common polytope we will examine in this section is the *cross-polytope*, also called the *orthoplex*. It is the simplest nontrivial polytope that is symmetric about the origin, as shown by Figure 3 for the case of $d = 2$.

Definition 3.7. The d-cross-polytope, denoted by $\Diamond_d$, is the polytope whose vertices are the d unit vectors $\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_d$ and their negatives:

$$\begin{aligned}\Diamond_d &\coloneqq \operatorname{conv}\{\pm\mathbf{e}_1, \pm\mathbf{e}_2, \dots, \pm\mathbf{e}_d\}\\ &= \{(x_1, x_2, \dots, x_d) \in \mathbb{R}^d : |x_1| + |x_2| + \cdots + |x_d| \le 1\}.\end{aligned}$$

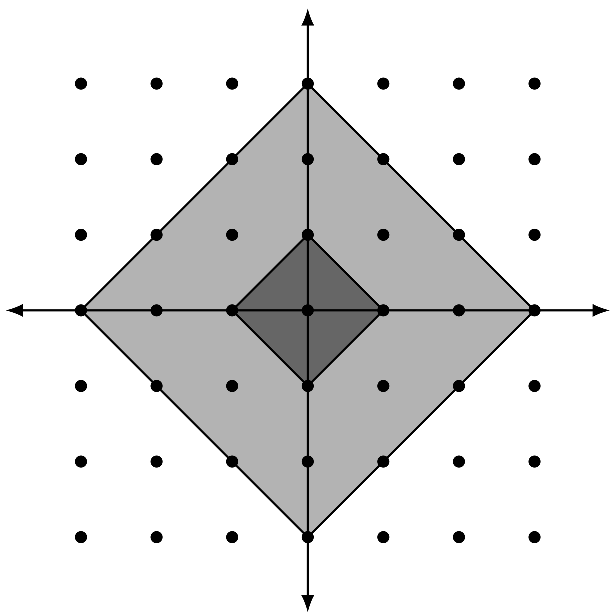

Figure 3. $\Diamond_2$ and $3\Diamond_2$ on the $\mathbb{Z}^2$ lattice.

The lattice-point enumerator of the d-cross-polytope is given by the following theorem.

Theorem 3.8. *The lattice-point enumerator of the d-cross-polytope is*

$$L_{\Diamond_d}(t) = \sum_{k=0}^{d} 2^k \binom{d}{k}\binom{t}{k}.$$

Proof. Since a given lattice point $(x_1, x_2, \dots, x_d)$ in $t\Diamond_d$ satisfies

$$|x_1| + |x_2| + \cdots + |x_d| \le t, \tag{3.3}$$

$L_{\Diamond_d}(t)$ counts the number of integer solutions to (3.3).

Using casework, we consider the case where exactly k of the x_i's are nonzero. Because all of the x_i's are symmetric to each other, we can WLOG assume that $x_1, x_2, \dots, x_k \ne 0$ and $x_{k+1}, x_{k+1}, \dots, x_d = 0$. Additionally, since all of the x_i's are inside absolute values, we can without loss of generality assume that $x_1, x_2, \dots, x_k > 0$.

With these assumptions, (3.3) becomes

$$x_1 + x_2 + \cdots + x_k \leq t \tag{3.4}$$

where $x_1, x_2, \ldots, x_k \in \mathbb{Z}_{>0}$. Note that (3.4) is very similar to (3.1) for the standard d-simplex. We can transform it into a inequality we are already familiar with by making the substitution $x_i = x_i' + 1, x_i' \in \mathbb{Z}_{\geq 0}$ for all i. (3.4) then becomes

$$\begin{aligned}(x_1' + 1) + (x_2' + 1) + \cdots + (x_k' + 1) &\leq t \\ x_1' + x_2' + \cdots + x_k' &\leq t - k.\end{aligned}$$

Now, using the technique of adding a slack variable $x_{k+1}' \in \mathbb{Z}_{\geq 0}$, we get

$$x_1' + x_2' + \cdots + x_k' + x_{k+1}' = t - k. \tag{3.5}$$

Applying the Stars and Bars formula, we get that (3.5) has $\binom{(t-k)+(k+1)-1}{(k+1)-1} = \binom{t}{k}$ solutions.

To account for the two assumptions made, we note that there are $\binom{d}{k}$ ways of selecting which x_i's were nonzero, and that there are 2^k ways to assigning signs to each selected x_i since each x_i can be either positive or negative. Multiplying everything together gives that there are

$$2^k \binom{d}{k}\binom{t}{k}$$

solutions to (3.3) in the case that k of the x_i's are nonzero. Summing over all possible $0 \leq k \leq d$ gives the desired formula for $L_{\Diamond_d}(t)$. ■

4. Ehrhart Series

Oftentimes in lattice-point enumeration, instead of analyzing the lattice-point enumerator function directly, it can be more useful to analyze its *generating function* — the polytope's *Ehrhart series*.

Definition 4.1. The generating function of an infinite sequence $a_0, a_1, a_2, \ldots$ is the power series

$$f(z) = a_0 + a_1 z + a_2 z^2 + \cdots = \sum_{k \geq 0} a_k z^k.$$

Definition 4.2. The Ehrhart series of a polytope $\mathcal{P}$, denoted $\mathrm{Ehr}_{\mathcal{P}}(z)$, is the generating function of $L_{\mathcal{P}}(t)$ — precisely, of the infinite sequence $L_{\mathcal{P}}(0), L_{\mathcal{P}}(1), L_{\mathcal{P}}(2), \ldots$ — where

$$\mathrm{Ehr}_{\mathcal{P}}(z) := 1 + \sum_{t \geq 1} L_{\mathcal{P}}(t) z^t$$

or simply

$$\mathrm{Ehr}_{\mathcal{P}}(z) := \sum_{t \geq 0} L_{\mathcal{P}}(t) z^t$$

if using the convention that $L_{\mathcal{P}}(0) = 1$ for all polytopes $\mathcal{P}$.

As examples, the following are the Ehrhart series for $\square_d$ and Δ_d.

Example. The Ehrhart series of $\square_d$ is

$$\mathrm{Ehr}_{\square_d}(z) = \sum_{t \geq 0} L_{\square_d}(t) z^t = \sum_{t \geq 0} (t+1)^d z^t = \sum_{t \geq 1} t^d z^{t-1} = \frac{1}{z} \sum_{t \geq 1} t^d z^t.$$

Example. The Ehrhart series of Δ_d is

$$\mathrm{Ehr}_{\Delta_d}(z) = \sum_{t\geq 0} L_{\Delta_d}(t) z^t = \sum_{t \geq 0} \binom{d+t}{d} z^t.$$

However, in the case of Δ_d, we can actually find a simpler expression for $\mathrm{Ehr}_{\Delta_d}(z)$ by solving for it directly without first finding $L_{\Delta_d}(t)$.

Recall Equation (3.2):

$$x_1 + x_2 + \cdots + x_d + x_{d+1} = t,$$

and recall that $L_{\Delta_d}(t)$ equals the number of nonnegative integer solutions to (3.2).

Each of the x_i's can be any nonnegative integer, so they all have generating functions

$$\sum_{k\geq 0} z^k = 1 + z + z^2 + z^3 + \cdots.$$

The generating function of a sum is the product of the generating functions of each summand variable. As such, the generating function for $x_1 + x_2 + \cdots + x_d + x_{d+1}$ is

$$(1 + z + z^2 + \cdots)^{d+1}.$$

The coefficient of z^k in this generating function, when expanded fully, equals the number of ways to choose nonnegative integers $x_1, x_2, \ldots, x_d, x_{d+1}$ such that $x_1 + x_2 + \cdots + x_d + x_{d+1} = k$. Thus, $(1+z+z^2+\cdots)^{d+1}$ is the generating function of $L_{\Delta_d}(t)$ and the lattice-point enumerator of Δ_d.

To further simplify this, we use the formula for an infinite geometric series in place of the infinite sum:

$$(1 + z + z^2 + \cdots)^{d+1} = \left(\frac{1}{1-z}\right)^{d+1} = \frac{1}{(1-z)^{d+1}}.$$

This gives the following theorem.

Theorem 4.3. *The Ehrhart series of Δ_d is*

$$\mathrm{Ehr}_{\Delta_d}(z) = \sum_{t\geq 0} \binom{d+t}{d} z^t = \frac{1}{(1-z)^{d+1}}.$$

Since the coefficients of a polytope's Ehrhart series encodes the values of its lattice-point enumerator function, we can learn a lot about a polytope by simply analyzing its Ehrhart series. We will see this in use in the later sections.

For a more in-depth introduction to generating functions, see [3].

5. Pick's Theorem

One of the most famous theorems involving lattice-point enumeration is *Pick's theorem*, published by Austrian mathematician Georg Alexander Pick in [5] in 1899.

Theorem 5.1 (Pick's theorem)**.** *Given a convex integral polygon $\mathcal{P}$, let the number of lattice points strictly interior to $\mathcal{P}$ be I, and let the number of lattice points on the boundary of $\mathcal{P}$ be B. Then, the formula*

$$A = I + \frac{B}{2} - 1$$

gives the area A of $\mathcal{P}$.

First, we will prove Pick's theorem using a traditional method. Then, we will derive a relationship between Pick's theorem and Ehrhart theory.

Before we begin to formally prove Pick's theorem, we first need to prove several lemmas that will serve as pieces for the main proof.

Lemma 5.2. *Given a convex integral polygon $\mathcal{P}$, it is always possible to decompose $\mathcal{P}$ into integral triangles (triangulation), that is, find a set of integral triangles whose interiors are disjoint and whose union is $\mathcal{P}$.*

Proof. Let the vertices of $\mathcal{P}$ be $V_1, V_2, \ldots, V_k$ in clockwise order. Since $\mathcal{P}$ is convex, we can then construct the segments $\overline{V_1V_3}, \overline{V_1V_4}, \ldots, \overline{V_1V_{k-1}}$, as shown in Figure 4.

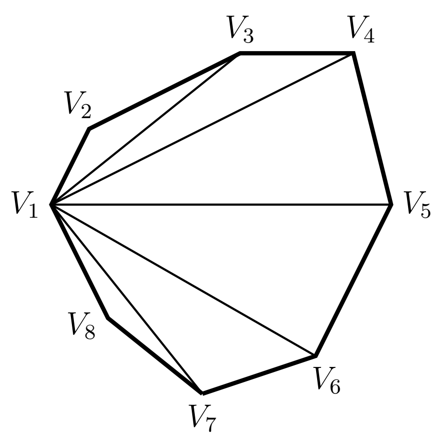

Figure 4. Triangulation of an octagon.

Then, let $\mathcal{T}_i = \triangle V_1V_iV_{i+1}$ for $2 \leq i \leq k-1$. Clearly, the $\mathcal{T}_i$'s only intersect on the edges $\overline{V_1V_3}, \overline{V_1V_4}, \ldots, \overline{V_1V_{k-1}}$, so their interiors are disjoint.

The union of the $\mathcal{T}_i$'s is obviously $\mathcal{P}$, since they completely cover $\mathcal{P}$, as Figure 4 shows.

Lastly, since we did not create any new vertices, the vertices of all of the $\mathcal{T}_i$'s are the vertices of $\mathcal{P}$, so the $\mathcal{T}_i$'s must be integral.

Therefore, we have a valid integral triangulation of $\mathcal{P}$, which proves the lemma. ■

Lemma 5.3. *Given a convex integral polygon $\mathcal{P}$, let a line intersecting the boundary of $\mathcal{P}$ at lattice points divide $\mathcal{P}$ into integral polygons $\mathcal{P}_1$ and $\mathcal{P}_2$. Then,*

(1) *if Pick's theorem holds for $\mathcal{P}_1$ and $\mathcal{P}_2$, it also holds for $\mathcal{P}$.*
(2) *if Pick's theorem holds for $\mathcal{P}$ and $\mathcal{P}_1$, it also holds for $\mathcal{P}_2$.*

Proof. Let A, I, B be the area of $\mathcal{P}$, the number of lattice points stricty inside $\mathcal{P}$, and the number of lattice points on the boundary of $\mathcal{P}$, respectively.

Similarly, let A_i, I_i, B_i be the area of $\mathcal{P}_i$, the number of lattice points stricty inside $\mathcal{P}_i$, and the number of lattice points on the boundary of $\mathcal{P}_i$, respectively, for $i = 1, 2$.

Obviously,

$$A = A_1 + A_2. \tag{5.1}$$

Then, if we let the number of lattice points on the common edge of $\mathcal{P}_1$ and $\mathcal{P}_2$ be E, we have

$$I = I_1 + I_2 + E - 2 \tag{5.2}$$

because the common edge is part of the interior or $\mathcal{P}$, but we have to subtract 2 to remove the two vertices of the common edge.

We also have

$$B = B_1 + B_2 - 2E + 2 \tag{5.3}$$

because the common edge is not part of the boundary of $\mathcal{P}$. Since it is included once in B_1 and once in B_2, we have to subtract it twice. Lastly, we have to add back 2 to account for the two vertices of the common edge which we removed.

In part (1), we assume that Pick's theorem holds for $\mathcal{P}_1$ and $\mathcal{P}_2$, so we have

$$A_1 = I_1 + \frac{B_1}{2} - 1,$$
$$A_2 = I_2 + \frac{B_2}{2} - 1.$$

This combined with (5.1), (5.1), and (5.3) gives

$$\begin{aligned} A &= A_1 + A_2 \\ &= I_1 + \frac{B_1}{2} - 1 + I_2 + \frac{B_2}{2} - 1 \\ &= I_1 + I_2 + \frac{B_1 + B_2}{2} - 2 \\ &= I_1 + I_2 + E - 2 + \frac{B_1 + B_2 - 2E + 2}{2} - 1 \\ &= I + \frac{B}{2} - 1, \end{aligned}$$

which proves part (1).

In part (2), we assume that Pick's theorem holds for $\mathcal{P}$ and $\mathcal{P}_1$, so we have

$$A = I + \frac{B}{2} - 1,$$
$$A_1 = I_1 + \frac{B_1}{2} - 1.$$

This combined with (5.1), (5.1), and (5.3) gives

$$\begin{aligned} A_2 &= A - A_1 \\ &= I + \frac{B}{2} - 1 - I_1 - \frac{B_1}{2} + 1 \\ &= I - I_1 + \frac{B - B_1}{2} \\ &= I_2 + E - 2 + \frac{B_2 - 2E + 2}{2} \\ &= I_2 + \frac{B_2}{2} - 1, \end{aligned}$$

which proves part (2). ■

Lemma 5.4. *Pick's theorem holds for all integral rectangles $\mathcal{R}$ whose sides are parallel to the axes.*

Proof. Let the width of $\mathcal{R}$ be w and the height of $\mathcal{R}$ be h. Then, without loss of generality let the bottom-left vertex of $\mathcal{R}$ be $(0, 0)$ and the top-right vertex of $\mathcal{R}$ be (w, h). Use the usual definitions of A, I, and B, so we have

$$A = wh. \tag{5.4}$$

The lattice points strictly inside $\mathcal{R}$ form a rectangular grid with bottom-left vertex $(1, 1)$ and top-right vertex $(w - 1, h - 1)$, so we have

$$I = (w - 1)(h - 1). \tag{5.5}$$

The horizontal edges of $\mathcal{R}$ each have length w, so they contain $w + 1$ lattice points each. Likewise, the vertical edges each have length h, so they contain $h + 1$ lattice points each. This gives that

$$B = 2(w + 1) + 2(h + 1) - 4 = 2w + 2h. \tag{5.6}$$

We need to subtract 4 since otherwise we would count the vertices twice.

Putting together (5.4), (5.5), and (5.6), we have

$$\begin{aligned} I + \frac{B}{2} - 1 &= (w - 1)(h - 1) + \frac{2w + 2h}{2} - 1 \\ &= wh - w - h + 1 + w + h - 1 \\ &= wh = A, \end{aligned}$$

as desired. ■

Lemma 5.5. *Pick's theorem holds for all integral right triangles $\mathcal{T}$ whose legs are parallel to the axes.*

Proof. Let the legs of $\mathcal{T}$ have lengths w and h. Then, without loss of generality let the vertices of $\mathcal{T}$ be $(0, 0)$, (w, h), and $(w, 0)$. Again, use the usual definitions of A, I, and B, so we have

$$A = \frac{wh}{2}. \tag{5.7}$$

Let H be the number of lattice points on the hypotenuse of $\mathcal{T}$. Then, if we use the same definition of rectangle $\mathcal{R}$ as in the proof of Lemma 5.4, we see that the number of internal lattice points of $\mathcal{T}$ is half the number of internal lattice points of $\mathcal{R}$ that do not lie on the hypotenuse of $\mathcal{T}$. This then gives

$$I = \frac{(w - 1)(h - 1) - H + 2}{2} \tag{5.8}$$

because we need to add 2 in the numerator to account for the vertices.

For the boundary, the horizontal and vertical legs contain $w + 1$ and $h + 1$ lattice points, respectively. The hypotenuse contains H lattice points by definition. Summing and subtracting 3 to account for the vertices gives

$$B = (w + 1) + (h + 1) + H - 3 = w + h + H - 1. \tag{5.9}$$

Putting together (5.7), (5.8), and (5.9) gives

$$\begin{aligned} I + \frac{B}{2} - 1 &= \frac{(w-1)(h-1) - H + 2}{2} + \frac{w + h + H - 1}{2} - 1 \\ &= \frac{(w-1)(h-1) - H + 2 + w + h + H - 1 - 2}{2} \\ &= \frac{wh - w - h + 1 + w + h - 1}{2} \\ &= \frac{wh}{2}, \end{aligned}$$

as desired. ■

With these lemmas, we can piece together the proof of Pick's theorem.

Proof of Theorem 5.1. Since $\mathcal{P}$ is convex, by Lemma 5.2, we can decompose $\mathcal{P}$ into a set of integral triangles the combine to form $\mathcal{P}$. Then, by part (1) of Lemma 5.3, if Pick's theorem holds for each of the triangles, then it must also hold for $\mathcal{P}$ because we only split $\mathcal{P}$ along its diagonals in the triangulation. Therefore, it suffices to proof Pick's theorem for integral triangles.

As a further simplification, notice that every integral triangle can be inscribed inside an integral rectangle whose sides are parallel to the coordinate axes, as shown in Figure 5. By part (2) of Lemma 5.3, this means that it suffices to prove Pick's theorem for integral rectangles whose sides are parallel to the axes and integral right triangles whose legs are parallel to the axes.

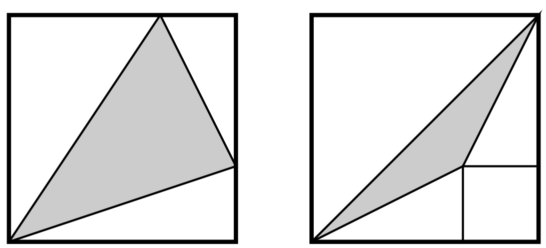

Figure 5. The two types of inscribed triangles in rectangles from [1].

Since we already know that Pick's theorem holds for those two types of polygons by Lemma 5.4 and Lemma 5.5, we can conclude that Pick's theorem must hold for any convex integral polygon, finishing the proof. ■

With Pick's theorem, we can actually derive the general form of the lattice-point enumerator for all convex integral polygons. We have the following theorem.

Theorem 5.6. *Given a convex integral polygon $\mathcal{P}$, let its area be A and let the number of lattice points on its boundary be B. Then,*

$$L_{\mathcal{P}}(t) = At^2 + \frac{B}{2}t + 1.$$

Proof. Consider $t\mathcal{P}$, the t^{th} dilate of $\mathcal{P}$. Let its area be A_t, the number of lattice points strictly in its interior be I_t, and the number of lattice points on its boundary be B_t.

By definition, we have

$$L_{\mathcal{P}}(t) = I_t + B_t. \tag{5.10}$$

Then, since $t\mathcal{P}$ is $\mathcal{P}$ under a dilation by a factor of t, we have that the area increases by a factor of t^2. This gives

$$A_t = At^2. \tag{5.11}$$

Next, suppose that $\mathcal{P}$ has n edges, and let them have $b_1, b_2, \ldots, b_n$ lattice points, respectively. This gives us

$$B = \sum_{k=1}^{n} b_k - n, \tag{5.12}$$

where we have subtracted n from the sum to correct for counting each vertex twice.

Continuing, let the edges of $t\mathcal{P}$ have $b_{t,1}, b_{t,2}, \ldots, b_{t,n}$ lattice points, in the same order as in $\mathcal{P}$. For a given $1 \leq i \leq n$, we have

$$b_{t,i} = tb_i - t + 1.$$

This is because dilating an edge by a factor of t is equivalent to joining together t copies of the original edge. Since there are $t-1$ points where the edges are joined together, we must subtract $t-1$ from tb_i to correct for overcounting. Summing together all of the $b_{t,i}$'s with correcting for overcounting and using (5.12) gives the following expression for B_t.

$$\begin{aligned}
B_t &= \sum_{k=1}^{n} b_{t,k} - n \\
&= \sum_{k=1}^{n} (tb_k - t + 1) - n \\
&= \sum_{k=1}^{n} tb_k - nt + n - n \\
&= t\sum_{k=1}^{n} b_k - nt = t\left(\sum_{k=1}^{n} b_k - n\right) = Bt. \qquad (5.13)
\end{aligned}$$

To finish, Pick's theorem combined with (5.10), (5.11), and (5.13) gives

$$\begin{aligned}
A_t &= I_t + \frac{B_t}{2} - 1, \\
A_t &= I_t + B_t - \frac{B_t}{2} - 1, \\
A_t &= L_{\mathcal{P}}(t) - \frac{B_t}{2} - 1, \\
L_{\mathcal{P}}(t) &= A_t + \frac{B_t}{2} + 1, \\
L_{\mathcal{P}}(t) &= At^2 + \frac{B}{2}t + 1,
\end{aligned}$$

concluding the proof. ■

6. Preparations for Main Results

Now that we have a deeper understanding of lattice-point enumerators and Ehrhart series, are ready to discuss why the lattice-point enumerator is called the *Ehrhart polynomial* in the first place.

Because all four of the lattice-point enumerators we have seen are polynomials in t, we might make the guess that the reason is that the lattice-point enumerator is always a polynomial for the classes of polytopes Ehrhart theory is interested in. The following theorem provides a precise answer.

Theorem 6.1 (Ehrhart's theorem)**.** *Given a convex integral polytope $\mathcal{P} \subset \mathbb{R}^d$, the lattice-point enumerator $L_{\mathcal{P}}(t)$ of $\mathcal{P}$ is a rational polynomial of degree d.*

This theorem was proved in 1962 by French mathematician Eugène Ehrhart in [2], who made extensive contributions to lattice-point enumeration. As such, the Ehrhart polynomial and Ehrhart theory are named in his honor.

As with Pick's Theorem, we will state several helpful lemmas before proving Theorem 6.1. We begin with the triangulation lemma for polytopes, a generalization of Lemma 5.2.

Definition 6.2. Let a *triangulation* of a convex polytope $\mathcal{P} \subset \mathbb{R}^d$ be a finite collection T of $(d-1)$-simplices such that

$$\mathcal{P} = \bigcup_{\Delta \in T} \Delta$$

and $\Delta_1 \cap \Delta_2$ is a common face of Δ_1 and Δ_2 for every $\Delta_1, \Delta_2 \in T$.

Theorem 6.3. *Given a convex integral polytope $\mathcal{P} \subset \mathbb{R}^d$, it is always possible to triangulate $\mathcal{P}$ using no new vertices.*

Proof outline. We will outline the main ideas of a proof. The full proof of this theorem can be found in [1, Theorem 3.1].

We can prove that a triangulation exists by constructing one. We begin by "lifting" $\mathcal{P}$ into $\mathbb{R}^{d+1}$. Letting the vertices of $\mathcal{P}$ be $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$, we then randomly choose $h_1, h_2, \ldots, h_n \in \mathbb{R}$ and construct a new polytope $\mathcal{Q} \subset \mathbb{R}^{d+1}$ such that

$$\mathcal{Q} = \operatorname{conv}\{(\mathbf{v}_1, h_1), (\mathbf{v}_2, h_2), \ldots, (\mathbf{v}_n, h_n)\}.$$

Then, consider the *lower hull* of $\mathcal{Q}$, the set of points $(x_1, x_2, \ldots, x_{d+1}) \in \mathcal{Q}$ such that there does not exist a point $(x_1, x_2, \ldots, x_{d+1} - x) \in \mathcal{Q}$ where $x > 0$.

Since the projection of the lower hull into $\mathbb{R}^d$ by removing the last coordinate of each point is $\mathcal{P}$, it suffices to prove that each face of the lower hull is a simplex and that $\Delta_1 \cap \Delta_2$ is a common face of Δ_1 and Δ_2 for the projections Δ_1, Δ_2 of any two faces of the lower hull. ■

Then, we need to introduce the concept of *cones* and *coning.*

Definition 6.4. A cone $\mathcal{K} \subset \mathbb{R}^d$ is a set of points of the form

$$\mathcal{K} = \{\mathbf{v} + \lambda_1 \mathbf{w}_1 + \lambda_2 \mathbf{w}_2 + \cdots + \lambda_n \mathbf{w}_n : \lambda_i \geq 0 \text{ for } 1 \leq i \leq n\}$$

where $\mathbf{v}, \mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{v}_n \in \mathbb{R}^d$, and there exists some hyperplane H such that $H \cap \mathcal{K} = \{\mathbf{v}\}$.

Let $\mathbf{v}$ be called the *apex* of $\mathcal{K}$ and the $\mathbf{w}_i$'s be called the *generators* of $\mathcal{K}$. Call $\mathcal{K}$ *integral* if $\mathbf{w}_i \in \mathbb{Z}^d$, and likewise for *rational.* Lastly, call $\mathcal{K}$ *simplicial* if its generators are linearly independent and $n = d$.

Definition 6.5. Let $\mathcal{P} \subset \mathbb{R}^d$ be a convex polytope, and let its vertices be $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$. Then, consider $\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_n \in \mathbb{R}^{d+1}$, where $\mathbf{w}_i = (\mathbf{v}_i, 1)$ for $1 \leq i \leq n$. With this, define the cone over $\mathcal{P}$ as

$$\operatorname{cone}(\mathcal{P}) \coloneqq \{\lambda_1 \mathbf{w}_1 + \lambda_2 \mathbf{w}_2 + \cdots + \lambda_n \mathbf{w}_n : \lambda_i \geq 0 \text{ for } 1 \leq i \leq n\}.$$

Continuing, we define the multivariate generating function of some set $S \subset \mathbb{R}^d$ called the *integer-point transform.*

Definition 6.6. Given a set $S \subset \mathbb{R}^d$, let the integer-point transform of S be

$$\sigma_S(\mathbf{z}) = \sigma_S(z_1, z_2, \ldots, x_d) \coloneqq \sum_{\mathbf{m} \in S \cap \mathbb{Z}^d} \mathbf{z}^{\mathbf{m}},$$

where for $\mathbf{a} = (a_1, a_2, \ldots, a_d)$ and $\mathbf{b} = (b_1, b_2, \ldots, b_d)$ we have $\mathbf{a}^{\mathbf{b}} = a_1^{b_1} a_2^{b_2} \cdots a_d^{b_d}$.

As an example, for some set $S \subset \mathbb{R}^d$, we can evaluate $\sigma_S(\mathbf{z})$ at $\mathbf{z} = (1, 1, \ldots, 1)$ to retrieve the number of lattice points in S:

Example. For a bounded $S \subset \mathbb{R}^d$, we have

$$\sigma_S(1, 1, \ldots, 1) = \sum_{\mathbf{m} \in S \cap \mathbb{Z}^d} 1 = \left| S \cap \mathbb{Z}^d \right|.$$

Now, we need the following theorem involving cones and their integer-point transforms.

Theorem 6.7. *Let $\mathcal{K}$ be an integral simplicial d-cone with generators $\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_d$. Then, for some $\mathbf{v} \in \mathbb{R}^d$, the integer-point transform of $\mathbf{v} + \mathcal{K}$ is*

$$\sigma_{\mathbf{v}+\mathcal{K}}(\mathbf{z}) = \frac{\sigma_{\mathbf{v}+\Pi}(\mathbf{z})}{(1 - \mathbf{z}^{\mathbf{w}_1})(1 - \mathbf{z}^{\mathbf{w}_2}) \cdots (1 - \mathbf{z}^{\mathbf{w}_d})},$$

where

$$\Pi \coloneqq \{\lambda_1 \mathbf{w}_1 + \lambda_2 \mathbf{w}_2 + \cdots + \lambda_d \mathbf{w}_d : 0 \leq \lambda_i < 1 \text{ for } 1 \leq i \leq d\}$$

is the fundamental parallelepiped *of $\mathcal{K}$.*

Proof. Let $\mathbf{m} \in (\mathbf{v} + \mathcal{K}) \cap \mathbb{Z}^d$ be a lattice point in $\mathbf{v} + \mathcal{K}$. By definition, we have

$$\mathbf{m} = \mathbf{v} + \lambda_1 \mathbf{w}_1 + \lambda_2 \mathbf{w}_2 + \cdots + \lambda_d \mathbf{w}_d \tag{6.1}$$

for some unique $\lambda_1, \lambda_2, \ldots, \lambda_d \geq 0$.

Then, we can decompose each of the λ_i's into its integer and fractional parts by letting $\lambda_i = \lfloor \lambda_i \rfloor + \{\lambda_i\}$ for all i. Then, (6.1) becomes

$$\mathbf{m} = (\mathbf{v} + \{\lambda_1\}\mathbf{w}_1 + \{\lambda_2\}\mathbf{w}_2 + \cdots + \{\lambda_d\}\mathbf{w}_d) + (\lfloor \lambda_1 \rfloor \mathbf{w}_1 + \lfloor \lambda_2 \rfloor \mathbf{w}_2 + \cdots + \lfloor \lambda_d \rfloor \mathbf{w}_d). \tag{6.2}$$

We can set

$$\mathbf{p} = \mathbf{v} + \{\lambda_1\}\mathbf{w}_1 + \{\lambda_2\}\mathbf{w}_2 + \cdots + \{\lambda_d\}\mathbf{w}_d,$$

and note that $\mathbf{p} \in (\mathbf{v} + \Pi) \cap \mathbb{Z}^d$ because both $\mathbf{m}$ and $\lfloor \lambda_1 \rfloor \mathbf{w}_1 + \lfloor \lambda_2 \rfloor \mathbf{w}_2 + \cdots + \lfloor \lambda_d \rfloor \mathbf{w}_d$ are integral, and $\{\lambda_i\} < 1$ for all i.

Therefore, we can rewrite (6.2) as

$$\mathbf{m} = \mathbf{p} + k_1\mathbf{w}_1 + k_2\mathbf{w}_2 + \cdots + k_d\mathbf{w}_d \tag{6.3}$$

for some unique $\mathbf{p} \in (\mathbf{v} + \Pi) \cap \mathbb{Z}^d$ and some unique $k_1, k_2, \ldots, k_d \in \mathbb{Z}_{\geq 0}$. From (6.3), we have that the generating function whose coefficients are all of the $\mathbf{m} \in (\mathbf{v} + \mathcal{K}) \cap \mathbb{Z}^d$ is

$$\begin{aligned}\left(\sum_{\mathbf{p}\in(\mathbf{v}+\Pi)\cap\mathbb{Z}^d} \mathbf{z}^{\mathbf{p}}\right)\left(\sum_{k_1\geq 0} \mathbf{z}^{k_1\mathbf{w}_1}\right)\cdots\left(\sum_{k_d\geq 0} \mathbf{z}^{k_d\mathbf{w}_d}\right) &= \sigma_{\mathbf{v}+\Pi}(\mathbf{z})\left(\frac{1}{1-\mathbf{z}^{\mathbf{w}_1}}\right)\cdots\left(\frac{1}{1-\mathbf{z}^{\mathbf{w}_1}}\right) \\ &= \frac{\sigma_{\mathbf{v}+\Pi}(\mathbf{z})}{(1-\mathbf{z}^{\mathbf{w}_1})(1-\mathbf{z}^{\mathbf{w}_2})\cdots(1-\mathbf{z}^{\mathbf{w}_d})}.\end{aligned}$$

■

For the last piece of the puzzle we need before we can prove Ehrhart's theorem, we have the following theorem.

Theorem 6.8. *For a convex integral polytope $\mathcal{P} \subset \mathbb{R}^d$, its Ehrhart series can be written as*

$$\mathrm{Ehr}_{\mathcal{P}}(z) = \sigma_{\mathrm{cone}(\mathcal{P})}(1, 1, \ldots, 1, z).$$

Proof. To begin, we create cone($\mathcal{P}$), the cone over $\mathcal{P}$. Note that the intersection of cone($\mathcal{P}$) and the hyperplane $x_{d+1} = 1$ is $\mathcal{P}$ itself. More generally, the intersection of cone($\mathcal{P}$) and the hyperplane $x_{d+1} = t$ for some positive integer t is $t\mathcal{P}$.

Now, consider $\sigma_{\mathrm{cone}(\mathcal{P})}$. We can decompose cone($\mathcal{P}$) into layers, where each layer is the intersection of $\mathcal{P}$ and a hyperplane of the form $x_{d+1} = t$ for some positive integer t. This means that we can rewrite $\sigma_{\mathrm{cone}(\mathcal{P})}$ in terms of the $\sigma_{t\mathcal{P}}$'s. Because the points in layer t all have $x_{d+1} = t$ and all are in $t\mathcal{P}$, we have

$$\begin{aligned}\sigma_{\mathrm{cone}(\mathcal{P})}(z_1, z_2, \ldots, z_{d+1}) &= 1 + \sigma_{\mathcal{P}}(z_1, z_2, \ldots, z_d)z_{d+1} + \sigma_{2\mathcal{P}}(z_1, z_2, \ldots, z_d)z_{d+1}^2 + \cdots \\ &= 1 + \sum_{t\geq 1} \sigma_{t\mathcal{P}}(z_1, z_2, \ldots, z_d)z_{d+1}^t.\end{aligned}$$

Using the fact that $\sigma_{\mathcal{P}}(1, 1, \ldots, 1) = \left|\mathcal{P} \cap \mathbb{Z}^d\right| = L_{\mathcal{P}}(1)$, this gives

$$\begin{aligned}\sigma_{\mathrm{cone}(\mathcal{P})}(1, 1, \ldots, 1, z_{d+1}) &= 1 + \sum_{t\geq 1} \sigma_{t\mathcal{P}}(1, 1, \ldots, 1)z_{d+1}^t \\ &= 1 + \sum_{t\geq 1} L_{\mathcal{P}}(t)z_{d+1}^t \\ &= \mathrm{Ehr}_{\mathcal{P}}(z_{d+1}),\end{aligned}$$

which completes the proof. ■

7. Main Results

Finally, we are ready to prove Ehrhart's theorem.

Proof of Theorem 6.1. Given a convex integral polytope $\mathcal{P} \subset \mathbb{R}^d$, we can triangulated it by Lemma 6.3. Because simplices in our triangulation intersect in lower-dimensional simplices, $L_{\mathcal{P}}(t)$ must be a sum and difference of lattice-point enumerators of simplices. Therefore, it suffices to prove Ehrhart's theorem for simplices.

Then, we use the result [7, Corollary 4.3.1] from *Enumerative Combinatorics*:

Lemma. *For some $f : \mathbb{Z}_{\geq 0} \to \mathbb{C}$ and some nonnegative integer d, the following are equivalent:*

(1) $\sum_{t\geq 0} f(t)z^t = \frac{g(z)}{(1-z)^{d+1}}$ *and* $\deg g \leq d$.
(2) f *is a polynomial of degree d if and only if $g(1) \neq 0$.*

By this lemma, it suffices to prove that

$$\mathrm{Ehr}_\Delta(z) = \frac{g(z)}{(1-z)^{d+1}} \tag{7.1}$$

for some simplex $\Delta \subset \mathbb{R}^d$, where g has degree at most d and $g(1) \neq 0$.

Since Δ is a d-simplex, it has exactly $d+1$ vertices, which we can call $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_{d+1}$. This means that $\mathrm{cone}(\Delta)$ is simplicial, and we can call its generators $\mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_{d+1}$. By Theorem 6.7, we have

$$\sigma_{\mathrm{cone}(\Delta)}(\mathbf{z}) = \frac{\sigma_\Pi(\mathbf{z})}{(1-\mathbf{z}^{\mathbf{w}_1})(1-\mathbf{z}^{\mathbf{w}_2})\cdots(1-\mathbf{z}^{\mathbf{w}_{d+1}})} \tag{7.2}$$

where Π is the fundamental parallelepiped of $\mathrm{cone}(\Delta)$.

Now, if we let $\mathbf{z} = (1,1,\dots,1,z_{d+1})$ and have $\mathbf{w}_i = (\mathbf{v}_i, 1) = (v_{i,1}, v_{i,2}, \dots, v_{i,d}, 1)$, we get that $\mathbf{z}^{\mathbf{w}_i} = (1^{v_{i,1}})(1^{v_{i,2}})\cdots(1^{v_{i,d}})(z_{d+1}^1) = z_{d+1}$. This means that (7.2) becomes

$$\sigma_{\mathrm{cone}(\Delta)}(1,1,\dots,1,z_{d+1}) = \frac{\sigma_\Pi(1,1,\dots,1,z_{d+1})}{(1-z_{d+1})(1-z_{d+1})\cdots(1-z_{d+1})} = \frac{\sigma_\Pi(1,1,\dots,1,z_{d+1})}{(1-z_{d+1})^{d+1}} \tag{7.3}$$

By Theorem 6.8, (7.3) becomes

$$\mathrm{Ehr}_\Delta(z_{d+1}) = \frac{\sigma_\Pi(1,1,\dots,1,z_{d+1})}{(1-z_{d+1})^{d+1}} \tag{7.4}$$

Now, because (7.4) matches (7.1) in form, we have that by the lemma, it remains to prove that $\sigma_\Pi(1,1,\dots,1,z_{d+1})$ is a polynomial of degree at most d and $\sigma_\Pi(1,1,\dots,1,1) \neq 0$. The latter is obvious because

$$\sigma_\Pi(1,1,\dots,1,1) = \left|\Pi \cap \mathbb{Z}^{d+1}\right|$$

and Π contains the origin.

For the former, recall the definition of the integer-point transform:

$$\sigma_S(\mathbf{z}) = \sum_{\mathbf{m}\in S\cap\mathbb{Z}^d} \mathbf{z}^{\mathbf{m}}.$$

We can let $S = \Pi$, $\mathbf{z} = (1,1,\dots,1,z_{d+1})$, and expand $\mathbf{m}$ as $(m_1, m_2, \dots, m_{d+1})$, so we have

$$\begin{aligned}\sigma_\Pi(1,1,\dots,1,z_{d+1}) &= \sum_{\mathbf{m}\in\Pi\cap\mathbb{Z}^d} (1,1,\dots,1,z_{d+1})^{(m_1,m_2,\dots,m_{d+1})} \\ &= \sum_{\mathbf{m}\in\Pi\cap\mathbb{Z}^d} (1^{m_1})(1^{m_2})\cdots(1^{m_d})(z_{d+1})^{m_{d+1}} \\ &= \sum_{\mathbf{m}\in\Pi\cap\mathbb{Z}^d} (z_{d+1})^{m_{d+1}}.\end{aligned}$$

Therefore, it remains to prove that the $m_{d+1} \leq d$ for every $\mathbf{m} \in \Pi \cap \mathbb{Z}^d$.

From the definition of the fundamental parallelepiped, we have

$$\Pi = \{\lambda_1\mathbf{w}_1 + \lambda_2\mathbf{w}_2 + \cdots + \lambda_{d+1}\mathbf{w}_{d+1} : 0 \le \lambda_i < 1 \text{ for } 1 \le i \le d+1\}.$$

Because $\mathbf{w}_i = (\mathbf{v}_i, 1)$ for all i, for any $\mathbf{m} = (m_1, m_2, \ldots, m_{d+1}) \in \Pi \cap \mathbb{Z}^d$ we have

$$m_{d+1} = \lambda_1 + \lambda_2 + \cdots + \lambda_{d+1}$$

where $0 \le \lambda_i < 1$ for all i. This gives

$$m_{d+1} < 1 + 1 + \cdots + 1 = d + 1.$$

Since m_{d+1} must be an integer, it is at most d, implying that the degree of $\sigma_\Pi(1, 1, \ldots, 1, z_{d+1})$ is at most d. This concludes our proof of Ehrhart's theorem. ■

Regarding Ehrhart polynomials, we also have the following important theorem that builds upon Ehrhart's theorem.

Theorem 7.1. *For a given convex integral polytope $\mathcal{P} \subset \mathbb{R}^d$, let its Ehrhart polynomial be*

$$L_{\mathcal{P}}(t) = a_d t^d + a_{d-1}t^{d-1} + \cdots + a_1 t + 1.$$

Then, a_d equals the volume of $\mathcal{P}$.

Proof. In higher dimensions, the volume of a polytope $\mathcal{P} \subset \mathbb{R}^d$, denoted by $\operatorname{vol}\mathcal{P}$, can be thought of as the number of unit d-cubes needed to completely tile $\mathcal{P}$. As such, a rough approximation of $\operatorname{vol}\mathcal{P}$ can be found by simply taking the number of unit d-cubes in $\mathcal{P}$.

This approximation can be improved by reducing the size of the cubes. If we are tiling $\mathcal{P}$ with d-cubes of side length s (which then intuitively have volume s^d), then the approximation becomes the amount of smaller cubes in the tiling multiplied by the volume of each small cube.

If we take the limit as the side length of the d-cubes approaches 0, we approach the precise value of $\operatorname{vol}\mathcal{P}$. This is equivalent to counting the number of lattice points inside $\mathcal{P}$ on with a smaller and smaller lattice. As such, we can define

$$\operatorname{vol}\mathcal{P} \coloneqq \lim_{t\to\infty} \frac{1}{t^d} \left| \mathcal{P} \cap \left(\frac{1}{t}\mathbb{Z}\right)^d \right|.$$

Since shrinking the lattice by a factor of t is equivalent to expanding $\mathcal{P}$ by a factor of t, we can rewrite this definition as

$$\operatorname{vol}\mathcal{P} \coloneqq \lim_{t\to\infty} \frac{1}{t^d} \left| t\mathcal{P} \cap \mathbb{Z}^d \right| = \lim_{t\to\infty} \frac{1}{t^d} L_{\mathcal{P}}(t).$$

Using this definition, we have

$$\begin{aligned} \operatorname{vol}\mathcal{P} &= \lim_{t\to\infty} \frac{a_d t^d + a_{d-1}t^{d-1} + \cdots + a_1 t + 1}{t^d} \\ &= \lim_{t\to\infty} (a_d + a_{d-1}t^{-1} + \cdots + a_1 t^{-d+1} + t^{-d}) \\ &= a_d. \end{aligned}$$

■

Another important result relates the Ehrhart polynomials of a polytope $\mathcal{P}$ and its interior, $\mathcal{P}^\circ$.

The definition of $L_\mathcal{P}(t)$ makes intuitive sense for nonnegative values of t. However, since it is possible to evaluate $L_\mathcal{P}(t)$ at the negative integers, we can ask what happens when we do so.

The following remarkable result was proved by British mathematician Ian G. Macdonald in 1971.

Theorem 7.2 (Ehrhart-Macdonald Reciprocity). *Given a convex integral polytope $\mathcal{P} \subset \mathbb{R}^d$, evaluating $L_\mathcal{P}(t)$ at the negative integers yields*

$$L_\mathcal{P}(-t) = (-1)^d L_{\mathcal{P}^\circ}(t).$$

Recall Theorem 3.4, where we found the Ehrhart polynomials of the unit d-cube and its interior:

$$L_{\square_d}(t) = (t+1)^d,$$
$$L_{\square_d^\circ}(t) = (t-1)^d.$$

Using Ehrhart-Macdonald Reciprocity, we can find $L_{\square_d^\circ}(t)$ directly from $L_{\square_d}(t)$:

Example. Evaluating $L_{\square_d}(t)$ at the negative integers gives

$$\begin{aligned} L_{\square_d}(-t) &= (-1)^d L_{\square_d^\circ}(t), \\ L_{\square_d^\circ}(t) &= (-1)^d \cdot L_{\square_d}(-t), \\ L_{\square_d^\circ}(t) &= (-1)^d \cdot (-t+1)^d, \\ L_{\square_d^\circ}(t) &= (-(-t+1))^d, \\ L_{\square_d^\circ}(t) &= (t-1)^d, \end{aligned}$$

as expected.

The full original proof of Ehrhart-Macdonald Reciprocity can be found in [4], and an alternative proof by Steven V Sam can be found in [6].

8. Ehrhart Positivity

We conclude with an open field of research in Ehrhart theory — *Ehrhart positivity*. A convex integral polytope $\mathcal{P}$ is said to have Ehrhart positivity or be *Ehrhart positive* if $L_\mathcal{P}(t)$ has all positive coefficients. This gives the central question of this field of research:

Question 8.1. *Which classes of convex integral polytopes have Ehrhart positivity?*

It turns out that many simple classes of polytopes have Ehrhart positivity, including the unit d-cube and the standard d-simplex. We have the following theorems.

Theorem 8.2. *The unit d-cube $\square_d$ is Ehrhart positive.*

Proof. By Theorem 3.4, we have $L_{\square_d}(t) = (t+1)^d$. Since $(t+1)$ has all positive coefficients and d is a positive integer, $(t+1)^d$ clearly has positive coefficients as well. ■

Theorem 8.3. *The standard d-simplex Δ_d is Ehrhart positive.*

Proof. By Theorem 3.6, we have $L_{\Delta_d}(t) = \binom{d+t}{d}$. By the definition of a binomial coefficient, we have

$$\begin{aligned} \binom{d+t}{d} &= \frac{(d+t)!}{d! \cdot t!} \\ &= \frac{(d+t)(d+t-1)\cdots(t+1)}{d!}. \end{aligned} \tag{8.1}$$

Since $d!$ is just a positive constant, we can ignore the denominator of (8.1). Then, in the numerator, each factor is a linear binomial with positive coefficients (the constant term ranges from 1 to d). Therefore, $\binom{d+t}{d}$, written as a polynomial, must have positive coefficients. ■

Of course, there are many other classes of polytopes that are Ehrhart positive. However, in general, it is significantly more difficult to prove or determine Ehrhart positivity for those classes of polytopes.

For example, the d-cross-polytope $\Diamond_d$ introduced earlier in this paper is Ehrhart positive. However, proving this fact requires us to first find $\mathrm{Ehr}_{\Diamond_d}(z)$, its Ehrhart series, as its Ehrhart polynomial $L_{\Delta_d}(t)$ is hard to work with.

Additionally, the methods needed to prove Ehrhart positivity for different Ehrhart positive classes of polytopes are often distinct from one another. This means that there is not yet any form of a standard procedure for determining or proving Ehrhart positivity. As such, Ehrhart positivity remains as a fascinating open area of research in Ehrhart theory.

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