Fractal dimension: a paradox of finite and infinite

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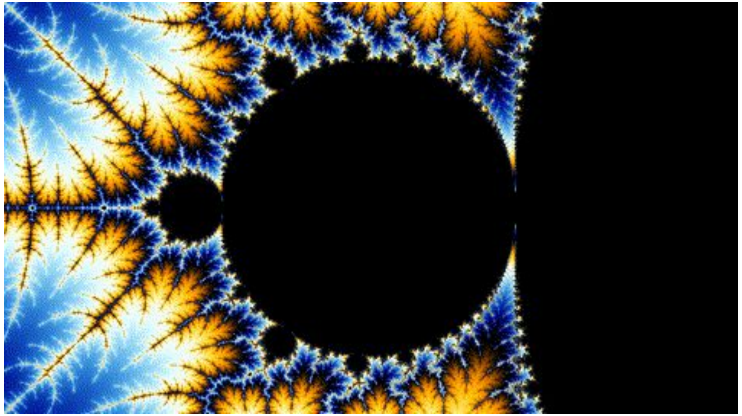


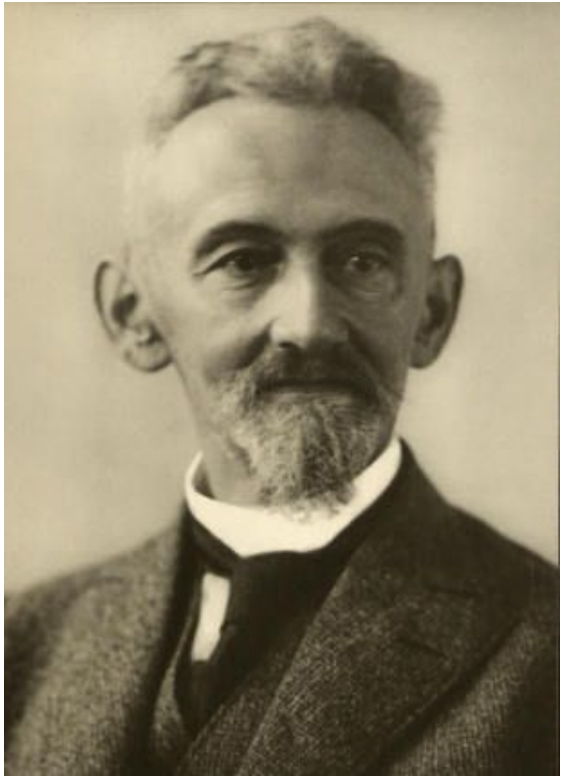
Area of Britain = 209331 km^2

...

But coastline Length of Britain?

Background

- First discovered in 1872 by Karl Weierstrass: called “Weierstrass function”
    - research on continuous but non-differentiable function
- Proved by the Hausdorff dimension
    - presented by Helix Hausdorff in 1918
- Examples of fractal dimension
    - Coastline paradox/ Coastline measure
    - Cantor Set



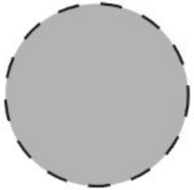
Open set

- **Definition**:

$$S = \bigcup I_i$$

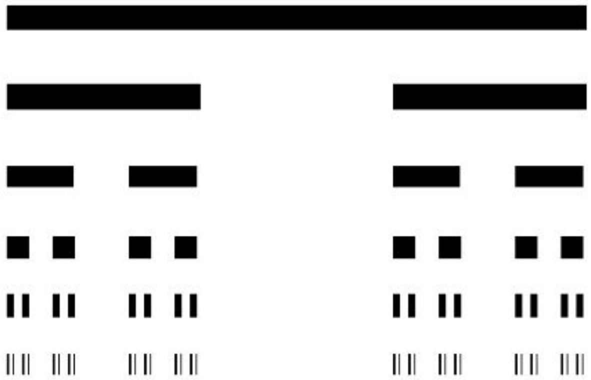
  - In the Euclidean space, open set *S* is a set composed of multiple open balls

- **Various dimensions**:
  - One dimensional → open interval: (a, b)
  - Two dimensional → open disk: D(r)
  - Three dimensional → open ball: B(r)



Cantor Set

- **Definition:**
  - topological space with a infinitely repeated structure by removing the middle-third part of the set and continuing that pattern with remaining segments
    - introduced by Georg Cantor in 1883



Cantor Set

- **Proposition 3.1:**

  Cantor set has length of complete zero

  - *l*(c) of [0,1] equals 1 (1-0=1)
  - For nth interval, starting from n=0...
    - 2^n intervals will be removed
    - with the length of $1/3^{n+1}$
  - Therefore, the following infinite geometric sequence would be constructed, resulting 1

$$\sum_{n=0}^{\infty} 2^n \cdot \frac{1}{3^{n+1}} = \frac{1}{3} \cdot \sum_{n=0}^{\infty} \frac{2^n}{3} = \frac{1}{3} \cdot \frac{1}{1-\frac{2}{3}} = 1$$

  - By *l*(c) -1 = 1-1 =0
    - It is proved that Cantor set has length of complete zero

Lebesgue measure

- **Definition**:

$$\lambda^*(E) = inf\left\{\sum_{k=1}^{\infty} l(I_k) : k \in \mathbb{R}, E \subset \bigcup_{k=1}^{\infty} I_k\right\}$$

  - In the open interval, the Lebesgue measure concerns the set different from the usual method of measure
    - l(a,b) = b-a
  - Instead, it considers the open interval to be sum of subsets with n-dimension

Fractal dimension

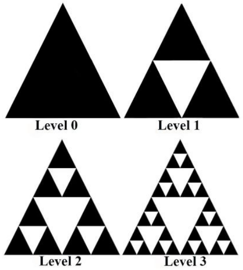
- **Fractal:**
  - infinitely continuous pattern with the property of self-similarity with certain scales
  - Mathematical notation:
    - set E be the whole fractal set
    - $(S_i)^n_{i=1}$ be the n-separated, self-similar subsets of E with the scale r<1

$$E = \bigcup S_i(E)$$

$$\frac{log n}{log \frac{1}{r}}$$

Fractal dimension

- **Example: Sierpinski triangle**
  - starts from the solid triangle
  - for each step, the triangle separates into 3 triangles with the length ratio of 0.5



  - Therefore, following the equation, the dimension of Sierpinski triangle is approx. 1.6.

$$\frac{log3}{log2} \approx 1.6$$

Hausdorff measure

- **Definition:**

$$H(S)^d_\delta = \inf\{\sum_{i=1}^{\infty}(diamUi)^d : \bigcup_{i=1}^{\infty} Ui \supseteq S, diamUi < \delta\}$$

  - Hausdorff measure depends on the concept of ‘cover’ which is a subset of the whole space and its diameter confined by a certain value delta.
    - The diameters of each covers vary

Hausdorff measure

- **Definition (continued):**

$$\lim_{\delta \to 0} H(S)_\delta^d = H(S)^d$$

  - After, by taking limit of delta to 0, the Hausdorff measure is computed.

Hausdorff measure/dimension

- **Proposition :**

$$if\ \delta_1 > \delta_2\ \ then\ H^d_{\delta 1}(S) < H^d_{\delta 2}(S)$$

  - Proof:
    - Case1: set of delta 1
      - maximum diameter of covers increases
        - there are more potential for the set of covers to be smaller once infimum is taken
    - Case2: set of delta 2
      - maximum diameter of covers decreases
        - there are less potential for the set of covers to be smaller once infimum is taken
    - Therefore, the small delta value has bigger Hausdorff measure because of the less potential to decrease

Hausdorff dimension

- **Definition:**

$$dim_H(S) := inf\ \{d \geq 0 : H^d(S) = 0\}$$

  - Starting the Hausdorff measure obtained by the previous slide, the Hausdorff dimension is defined by the minimum value of d (dimension) which makes Hausdorff measure 0.
  - In fractal structure...
    -

$$dim_H(E) = \frac{logn}{log\frac{1}{r}}$$

Coastal paradox

- **Definition:**
  - Paradox regarding the length of coastlines
    - coastlines indeed have finite length
    - But, they are often referred to have undefined/infinite length due to their infinitely repeated complex details
  - Other than conventional measure method:
    - Statistical self-similarity
      - coastlines are considered as fractal curve

Coastline measure

- **Method:**
  - First step
    - identify two points and construct the shortest line connecting two (would be a straight line) and label that line as ‘G’
  - Second step
    - Then, measure the dimension of the coastline by analyzing the breaking pattern of the sea coast and each line’s length
      - use the equation of fractal dimension to determine the value of D
  - Third step
    -

$$L(s) = M * G^{1-D}$$

Coastline Measure

- **Examples:**
  - west coast Britain's D (complex coastline): 1.25
  - South Africa coast's D (flat coastline): 1.02

Thank you