FRACTAL DIMENSION: A PARADOX OF INFINITE AND FINITE

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1. Abstract

The measure of a set is the infimum of the sum of the open subsets covering the set. For complicated or self-repeated sets, fractal measure and Hausdorff measure are used for analysis. Besides theoretical fractal structures, such as Cantor set, there are real-life applications of fractals, such as the measurement of a coastline, also known as the coastline paradox.

2. Introduction

Finite and infinite may not seem to coexist; however, in the fractal dimension, a paradox comes into an existence. Fractal dimension was first discovered by Karl Weierstrass in 1872 during his research on continuous but non-differential function; later, his function, so called, "Weierstrass function." The fractal dimension was developed by multiple researchers, such as Georg Cantor, Helge von Koch, and Waclaw Sierpinski, and later proved by the Hausdorff dimension which is a measure of the fractal dimension, first presented by Helix Hausdorff in 1918. Developed from the familiar concept of dimension, for instance, that a dot is zero dimension and the dimension extends as the dot constructs line or shape, the measure of theory is interpreted in an alternative way. The measure of the dimension is conducted by open sets, which consider interval unions of subsets. Also, they rely on the concepts of infimum and supremum, which may or may not be in the

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set itself. The measure of dimension starts from the basic concept, the Lebesgue measure takes the infimum of the union of open subsets covering the set. Once the dimension gets complex and the set gets self-repeated, more generalized methods, such as fractal dimension and Hausdorff dimension, are utilized; they often enables the measure of the non-integer dimension in complex objects. For instance, a one-dimensional object such as a curve can have more than one dimension due to it's fractal structure. Hausdorff dimension tend to rely on the concept of 'cover', which is also a subset of the n-dimension space. Although the concept looks complicated, it is applicable to various fields in real life. One famous example is the coastline paradox. The coastline has a finite length, but due to its complexity and fine details (unusually large number of corners), it's considered to be immeasurable. Therefore, the fractal dimension is engaged in the computation to analyze the coastline's self similarity and potentially repeated or even self-repeated structure.

3. Definitions

3.1. **Open Set.**

In the metric space, open set follows the concept of sets, which is the union of of finite or infinite subsets. Assume that S is an open set and I_i is open balls of the open set S, the open set satisfies the following equation:

$$S = \bigcup I_i$$

It is called as an open interval in one dimension, disk in two dimension, and ball in three dimension or higher. They are noted as followings:

$$(a, b), D(r), B(r)$$

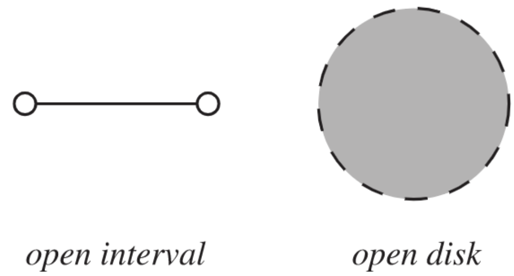


Figure 1. image of open set of interval and disk

Proposition 3.1.1 1.

$$|x_0 - x| < r$$

Assume that the open set has radius of r and centered at x_0. In any subsets x, none of the subsets exceeds or be equal to r.

Proof. Open Set doesn't let any subset be equal to or exceed its limit, therefore, any length between any subsets wouldn't exceed the total length of the open set, $l(S)$. □

Theorem3.1.1 2.

Infinite intersection of the open set:

$$\bigcap_{n=1}^{\infty} I(n)$$

$$n \epsilon \mathbb{N}$$

may or may not be open set

Proof. First assume that open set I(n) to be (-n,n). I(1) would be (-1, 1), I(2) would be (-2, 2), I(3) would be (-3, 3), and as n goes to infinity, the range of the set increases infinitely. Therefore, the intersection of the whole set:

$$\bigcap_{n=1}^{\infty}(-n,n)$$

would be (-1, 1), which is an open set. Up to this point, an infinite intersection of the open set seems to be open set.

However, let's assume that set I(n) to be $(-\frac{1}{n},\frac{1}{n})$. Different from the previous set, the range of the set would gradually decrease; I(1) would be (-1,1), I(2) would be $((-\frac{1}{2},\frac{1}{2})$, and I(3) would be $(-\frac{1}{3},\frac{1}{3})$, continuing infinitely. Once the set goes to infinite, the value of:

$$\lim_{n\to\infty}\left(-\frac{1}{n},\frac{1}{n}\right)$$

would be a single value of 0, not being an open set. Therefore, the value of:

$$\bigcap_{n=1}^{\infty}(-\frac{1}{n},\frac{1}{n})$$

would be 0, proving that an infinite intersection of the open set doesn't have to be open set.

□

3.2. **Lebesgue Measure.**

Assume that there is a open set I with range of (a,b). The usual measure of the set is written as $l(I) = b - a$. However, in the concept of Lebesgue Measure, the measure goes beyond the simple length but also considers the subsets of the whole set with n-dimension. For instance, the measure of the point in the certain interval according to the Lebesgue Measure is 0.

Let E be a subset of $\mathbb{R}$ and I_k be a sequence open intervals (open rectangles) that cover E. Then the definition of the Lebesgue Measure for any subset E satisfies the below:

$$\lambda^*(E) = inf\left\{\sum_{k=1}^{\infty} l(I_k) : k \in \mathbb{R}, E \subset \bigcup_{k=1}^{\infty} I_k\right\}$$

The infimum of the union of the length of the subsets in the interval indicates the measure of the set.

The Lebesgue Measure is the very basic concept of the measure of the space, but once the space gets complex and dimension increases, the more general concept called "Hausdorff measure and dimension" is used. This concept would be defined in the subsection 3.3.

3.3. **Fractal Dimension.**

A fractal is an infinitely continuous pattern that self-similarity continues with different scales.

The common perception regarding dimension denotes that dimensions are usually integer, for instance line is a one dimension. However, the fractal dimension breaks that perception and allows a rational dimension to exist. For instance, even though the object only consists of lines, it can have a rational dimension bigger than one.

Let $(S_i)_{i=1}^n$ be the n-separated self similarities with the ratio $r < 1$. Also, define the fractal set E to be a union of those self similar components:

$$E = \bigcup S_i(E)$$

Under this condition, dimension E satisfies the following equation:

$$\frac{log n}{log \frac{1}{r}}$$

Below is the famous example of the fractal structure: Sierpinski triangle

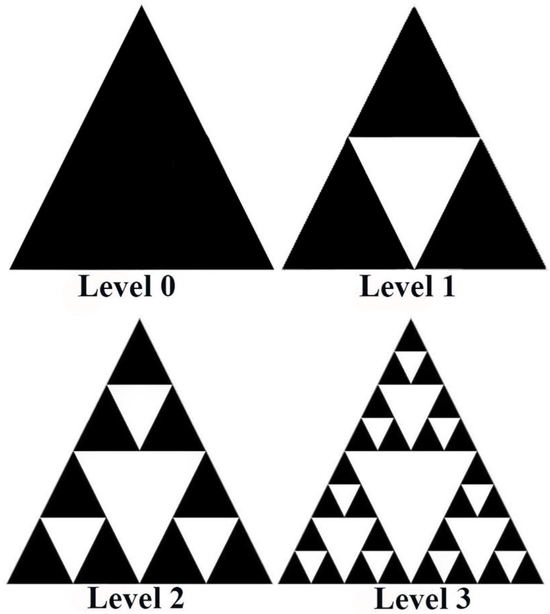


Figure 2. Sierpinski triangle

Once Sierpinski triangle goes through the each step, it separates into 3 triangles with the ratio of $\frac{1}{2}$. Therefore, the dimension the fractal structure follows the equation below:

$$\frac{log3}{log2} \approx 1.6$$

Therefore, Sierpinski triangle has a 1.6 dimension.

3.4. **Hausdorff Dimension.**

Assume that S be a subset of the metric space X and d ranges: $(0, \infty]$

$$H(S)^d_\delta = \inf\{\sum_{i=1}^{\infty} (diamUi)^d : \bigcup_{i=1}^{\infty} Ui \supseteq S, diamUi < \delta\}$$

Let S be a metric space and Ui be a subset whose union which i ranges from 1 to infinite and its d power covers the S; here, d indicates the dimension. The diameter of the cover Ui is confined under the maximum value ; the diameter of each covers can differ under the value , for instance the set a perfect fit that covers S, while some other sets are so immense that only has S as a partial set. Therefore, the infimum makes the set of Ui more suitable to the S by taking the minimum of the sets of covers within the range of dimension that contains the S.

$$\lim_{\delta \to 0} H(S)^d_\delta = H(S)^d$$

By taking limit of delta to 0, the Hausdorff measure is computed.

$$dim_H(S) := inf \left\{ d \geq 0 : H^d(S) = 0 \right\}$$

For the Hausdorff dimension, its value is the minimum of the values of d that makes the Hausdorff measure as 0.

Theorem3.4.1 1.

If the set E satisfies the following condition:

$$E = \bigcup S_i(E)$$

where $(Si)^n_{i=1}$is n-separated, self-similar subset of E with ratio $r < 1$

Then the Hausdorff dimension of E satisfies the following equation:

$$dim_H(E) = \frac{logn}{log\frac{1}{r}}$$

Proposition 3.4.1 3.

$$if\ \delta_1 > \delta_2\ then\ H^d_{\delta 1}(S) < H^d_{\delta 2}(S)$$

Proof. The bigger value means that the maximum of diameters of the covers increases which means there are more potential for the sets of the powers of the covers to be only smaller once infimum is taken. In reverse, the smaller value means that the maximum size of the covers decreases, indicating less potential for the sets to be smaller once infimum is taken. Therefore, the Hausdorff measure works in an inverse mechanic with the value of δ.

The proposition seems complicated with complex notations, but it can be applied to the familiar situations.

For the one-dimension example, let's consider two coastlines which will be discussed again later. One is the coastline of the Great Britain and the other is the coastline of South Africa. While South Africa's coastline is flat as a straight line, that of Britain is famous for its complicated and fine structure, often being a research subject for coastal paradox. The value of δ indicates the length of fractured line segments covering the coastline, which is smaller for Britain's coastline than for South Africa's. Therefore, according to the proposition, because $\delta_{Britain}$ is smaller than $\delta_{SouthAfrica}$, $dim_H(SouthAfrica) < dim_H(Britain)S$ should be satisfied; according to Mandelbrot, the Hausdorff dimension of Britain's coast line is 1.25 and South Africa's coastline is 1.02, corresponding to the proposition.

For the two-dimension example, let's consider Broccoli and Cauliflower. They might seem irrelevant to the Hausdorff dimension, but their surface can be considered as fractal structures with repeated small branches. According to San-Hoon Kim's research, the Broccoli's surface has the Hausdorff dimension of 2.7 and Cauliflower's surface has Hausdorff dimension of 2.8. This result implies that $\delta_{Broccoli} > \delta_{Cauliflower}$; in other words, Cauliflower has more complicated structure with smaller branches.

□

Proposition 3.4.2 4.

$$H^x(S) < \infty \rightarrow H^d(S) = 0 \; for \; (d > x)$$

$$H^x(S) > 0 \rightarrow H^d(S) = \infty \; for \; (d < x)$$

Proof. Since the radius of $diam_H(S)$ is confined under the value of δ, the following works:

$$diam_H(S)^d = r^d = r^{d-x} \cdot r^x < \delta^{d-x} \cdot r^x$$

Therefore, the below is satisfied.

$$H^d_\delta(S) \leq \delta^{d-x} \cdot H^d_\delta(S)$$

Earlier, we assumed the condition, $H^x(S) < \infty$ and $d - x > 0$. If we take limits on both side, it will yield the following Hausdorff dimension:

$$H^d(S) = \lim_{\delta \to 0} H(S)^d_\delta \leq \lim_{\delta \to 0} \delta^{d-x} \cdot H(S)^d_\delta = (\lim_{\delta \to 0} \delta^{d-x}) \cdot H(S)^d_\delta = 0$$

For the second proposition, the proof works in a similar way. In the equation above, under the new condition that $H^x(S) > 0$ and $d-x < 0$, the following Hausdorff dimension will be resulted:

$$H^d(S) = \lim_{\delta \to 0} H(S)^d_\delta \leq \lim_{\delta \to 0} \delta^{d-x} \cdot H(S)^d_\delta = (\lim_{\delta \to 0} \delta^{d-x}) \cdot H(S)^d_\delta = \infty$$

□

4. Fractals

4.1. **Koch Snowflake.**

Before we go to the coastline paradox, let's start with the fractal structure with a paradox of the coexistence of infinite and finite: Koch snowflake which was first introduced by Helge von Koch in 1904.

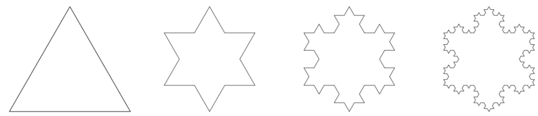


Figure 3. Koch snowflake through stage 0 to 3

Let Koch snowflake initiates from the unit equilateral triangle whose perimeter is 3 and area is $\frac{\sqrt{3}}{4}$. Then divide each side into 3 parts, by removing the middle one-third of each side and replacing them with a smaller equilateral triangle with side length of 1/3 the original triangle. New triangles with the length scale of $\frac{1}{3}$ are built around the original triangle. We repeat this process infinitely. Therefore, the new object at the stage one has perimeter of 4 and area of $\frac{\sqrt{3}}{4} + \frac{3\cdot(\frac{1}{3})^2\cdot\sqrt{3}}{4}$. In the stage two, again, the same pattern is applied to each side of the object, resulting in the new object having a perimeter of $\frac{16}{3}$ and area of$\frac{\sqrt{3}}{4} + \frac{3\cdot(\frac{1}{3})^2\cdot\sqrt{3}}{4} + \frac{\sqrt{3}\cdot 3\cdot(\frac{1}{3})^4\cdot 4}{4}$. The stage continues infinitely, and the same pattern is applied infinitely to each side of the object. Based on the pattern, in the nth stage, the Koch snowflake's parameter and area can be expressed as following functions:

$$\text{parameter} : P(n) = 3\cdot(\frac{4}{3})^n$$
$$\text{area} : A(n) = \frac{\sqrt{3}}{4} + \frac{\sqrt{3}}{4}\cdot\sum_{i=1}^{n}\frac{3}{4}\cdot\left(\frac{4}{9}\right)^n$$

If the limit when n approaches ∞ in order to compute the area and parameter of this infinitely continuing fractal, each function yields following results:

$$\lim_{n\to\infty} P(n) = \lim_{n\to\infty} 3\cdot(\frac{4}{3})^n = \infty$$
$$\lim_{n\to\infty} A(n) = \lim_{n\to\infty}\frac{\sqrt{3}}{20}\cdot(8 - 3\cdot(\frac{4}{9})^n) = \frac{2\sqrt{3}}{5}$$

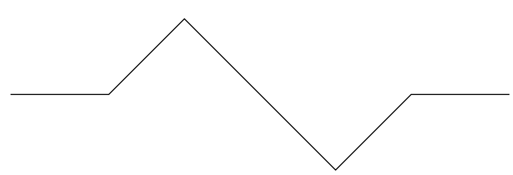
Here, it is proved that while Koch snowflake has finite area, it has an infinite or immeasurable parameter. Therefore, we can say that concepts of infinite and finite coexist in this fractal structure, which seems to be a paradox. This same notion exists in measuring certain coastlines, or does it?

4.2. **Coastal Paradox.**

There is a paradox regarding the length of coastlines. Coastlines indeed have finite length. However, they are often referred to have undefined or infinite length due to infinitely repeated complex details. Therefore, in order to measure their lengths, the method other than the conventional measure is required: statistical self-similarity. Because the perfectly fractal structure is seldom encountered in nature, the statistical self-similarity is used handling this kind of cases. In this case, coastlines are considered as fractal curve with a property of self-similarity with certain ratio of reduced scale.

The first step for the measurement is to identify two points and construct the shortest line connecting two, and identify it as G; G would have dimension of one like the figure below.

From now, dimension will be denoted as D. Depending on the coastline's fractal structure, in this case the breaking pattern of the sea coast and each line's length, D is determined based on the equation fractal dimension: $\frac{log n}{log \frac{1}{r}}$.



For the figure above, since the straight line is broken down into 6 small lines will the scale of $\frac{1}{4}$, therefore the figure will have D of $\frac{log 6}{log 4}$ which is approximately 1.3.

This method applies to the coastline measure to obtain the coastline's D value by breaking down the line G which connects two certain points of the coastline following the pattern of the coastline.

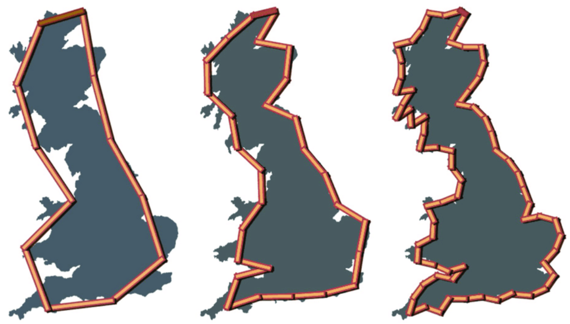


Figure 4. The steps of the application of the fractal dimension to the coastlines measure

For examples, as mentioned in the proposition 2 of the Hausdorff dimension, Because the west coast Britain has complicated and broken into small lines, it has Hausdorff dimension of approximately 1.25. In contrast, since South Africa's coastline tends to be flat, it has Hausdorff dimension of approximately 1.02.

After obtaining a value of dimension, the length of coastline satisfies the following equation:

$$L(s) = M * G^{1-D}$$

M is a positive constant and G and D are values explained in the previous paragraph.

4.3. **Cantor Set.**

Cantor Set was first introduced by Georg Cantor in 1883. It is a topological space with a infinitely repeated structure by removing the middle-third part of the set and continuing that pattern with remaining segments.

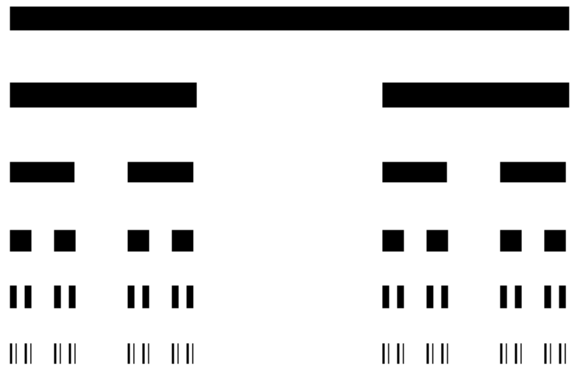


Figure 5. Image of the Cantor Set

Assume the interval of the set to be [0,1]. In the first step, the open subset of the middle third of the interval, $(\frac{1}{3}, \frac{2}{3})$, will be deleted, leaving a set $[0, \frac{1}{3}] \bigcup [\frac{2}{3}, 1]$.

In the next step, the pattern will be executed in remaining two sets. For the first segment, the open subset $(\frac{1}{9}, \frac{2}{9})$ is removed and in the second segment, the open subset $(\frac{7}{9}, \frac{8}{9})$ is removed. Therefore, the remaining set will be $[0, \frac{1}{9}] \bigcup [\frac{2}{9}, \frac{1}{3}] \bigcup [\frac{2}{3}, \frac{7}{9}] \bigcup [\frac{8}{9}, 1]$. This pattern continues infinitely and it can be expressed by the following equation:

$$C = \bigcap_{n=1}^{\infty} \bigcap_{k=0}^{3^{n-1}-1} \left(\left[0, \frac{3k+1}{3^n}\right] \bigcup \left[\frac{3k+2}{3^n}, 1\right] \right)$$

Proposition 4.2.1 5.

Cantor set's measure is complete zero

Proof. The proof will be based on the Cantor set of interval [0,1] and the length of the interval is equal to the l(C) which is 1. Starting from the n=0, for every nth term, 2^n intervals will be removed with the length of $\frac{1}{3^{n+1}}$. Therefore, this pattern would yield the following equation:

$$\sum_{n=0}^{\infty} 2^n \cdot \frac{1}{3^{n+1}}$$

The sequence is a typical example of infinitely continuing geometric sequence. Therefore, the sum of the sequence would be 1 by following proofs :

$$\sum_{n=0}^{\infty} 2^n \cdot \frac{1}{3^{n+1}} = \frac{1}{3} \cdot \sum_{n=0}^{\infty} \frac{2}{3}^n = \frac{1}{3} \cdot \frac{1}{1-\frac{2}{3}} = 1$$

Following the equation,

$$l(C) - 1 = 1 - 1 = 0$$

it is proved that the Cantor set has measure of zero. □

Proposition 4.2.2 6.

$$H^d(C) \approx 0.6$$

Proof. Cantor Set has Hausdorff dimension of approximately 0.6. Following the equation of the fractal dimension, $\frac{logn}{log\frac{1}{r}}$, since each line forms 2 lines with the ratio of $\frac{1}{3}$, the dimension of Cantor Set equals $\frac{log2}{log3}$, which yields 0.6. Even though it is a one-dimension figure, the dimension of Cantor Set is less than one. □

4.4. **Menger Sponge.**

Menger Sponge is a three dimensional fractal structure which is a generalization of the Cantor Set of the previous section. It was first discovered by Karl Menger in 1926, studying the concept of topological dimension.

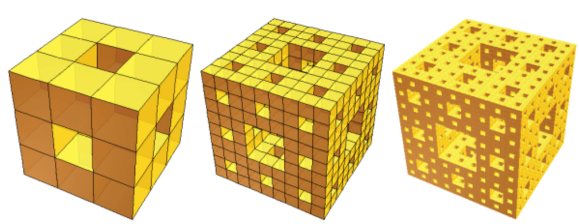


Figure 6. Image of Menger Sponge through stage 1 to 3

The construction begins from the solid unit cube. For the first stage, divide the cube into 27 identical cubes by dividing each surfaces into 9 identical squares, remove the center cube from each surfaces, and lastly remove the cube at the center of the large cube. After the first stage, the total of 20 cubes with length of $\frac{1}{3}$ will be left. This process iterates infinitely, applying the same method to each cubes. This pattern of the Menger Sponge can be defined by the following equation:

$$M := \bigcap_{n \in \mathbb{N}} M_n$$

First, set M to be a intersection of sets M_n. After, assume that M_0 equals the unit cube and no more than one of vector elements *i, j, k* equals to 1. Then M_{n+1} can be defined as following equation:

$$M_{n+1} := \left\{ (x, y, z) \, \epsilon \mathbb{R}^3 : \exists i, j, k \epsilon \, \{0, 1, 2\} : (3x - i, 3y - j, 3z - k) \, \epsilon M_n \right\}$$

Proposition 4.3.1 7.

$$H^d(M) \approx 2.7$$

Proof. Menger sponge has Hausdorff dimension of approximately 2.7. Following the equation of the fractal dimension, $\frac{log n}{log \frac{1}{r}}$, since each cube forms 20 self-similar cubes with the ratio of $\frac{1}{3}$, the dimension of Menger Sponge equals $\frac{log 20}{log 3}$, which yields 2.7. Even though it is a three-dimension figure, the dimension of Menger Sponge is less than three. □

Proposition 4.3.2 8.

$$\lim_{n\to\infty} V(M_n) \to 0$$

Proof. As the Menger Sponge iterates infinitely, its volume approaches to 0. For each interval, the volume decreases with the scale of $\frac{20}{27}$ because for the each cubes, 7 cubes are removed from the original 27 cubes. Therefore, $V(M_n) = (\frac{20}{27})^n$. Because the common ratio is smaller than 1, the volume goes to 0 as it n goes to infinite. □

5. Conclusion

The idea that a shape with a finite area may have an infinite perimeter seems fascinating to the curious mind. Hypothetically, even it none of the coastlines have infinite length. How can an island have an unusually large perimeter relative to it's perimeter? From my research and a learning experience, I believe there are certain areas in advanced science in fields such as astronomy or biology, whose progress go hand with this concept of measure, which leaves a great room available for future research open. I would like to contribute to this field by exploring applications in real life that haven't reached a conclusion yet and are considered open problems. Maybe I can find the Hausdorff dimension of the Korean Peninsula, my mother land. Or maybe I can find one for Malaysia, or Indonesia whose geography with multiple islands intrigued me. Also, not only for the coastlines, I can discover more of both theoretical and real-world fractals. And I hope, one day, I can make groundbreaking progress in this field.

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