The Generalized Stokes' Theorem

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July 9, 2022

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Introduction

Although the "generalized Stokes' theorem" may sound unfamiliar, most multivariable classes likely have introduced the original Stokes' theorem:

$$\oint_C \mathbf{F} \cdot \mathbf{T}\, ds = \iint_\sigma (\text{curl } \mathbf{F}) \cdot \mathbf{n}\, dS$$

A bit of history on the original Stokes' Theorem: the theorem was actually first developed by Lord Kelvin, who communicated the result to George Stokes in a letter (1850).

Introduction

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$$\oint_C \mathbf{F} \cdot \mathbf{T}\, ds = \iint_{\sigma} (\text{curl } \mathbf{F}) \cdot \mathbf{n}\, dS$$

This is actually the "fake" Stokes' theorem! The generalized Stokes' theorem tells us much more than the original, spanning the three classical theorems of vector calculus.

The Generalized Stokes' Theorem

$$\int_{\partial A} \omega = \int_{A} d\omega$$

where A is any compact-oriented k-dimensional manifold with boundary and ∂A is a $k-1$ dimensional manifold with the boundary orientation.

Note the similarities between the generalized Stokes' theorem and the Fundamental Theorem of Calculus (Newton-Leibniz formula):

$$f(b) - f(a) = \int_{a}^{b} f'(x)\, dx.$$

The Generalized Stokes' Theorem

$$\int_{\partial A} \omega = \int_A d\omega$$

where A is any compact-oriented k-dimensional manifold with boundary and ∂A is a $k-1$ dimensional manifold with the boundary orientation.

Due to the theorem spanning most general manifolds, many definitions are made to cement the idea of a "manifold" and "orientation" among other concepts.

Manifolds

Definition

A manifold is a topological space that is *locally* Euclidean.

- Surfaces are 2-dimensional manifolds
- "Locally euclidean"

The Earth as a manifold.

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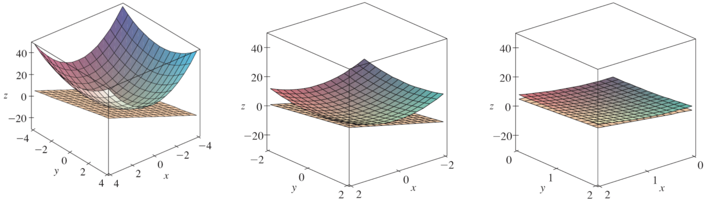

Manifolds

Definition

A manifold is a topological space that is *locally* Euclidean.

- Surfaces are 2-dimensional manifolds
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Defining manifolds of all kinds gets rather complicated.

Orientation

- Defining the "direction" by which we approach the manifold
- More complicated than counterclockwise vs. clockwise
- Why is the orientation of a manifold relevant?

When M lacks an orientation, reparametrizing M may cause the solution of the integral to change sign. The integral over a manifold is only *well-defined* when M has an orientation.

Orientation

- Defining the "direction" by which we approach the manifold
- More complicated than counterclockwise vs. clockwise
- Why is the orientation of a manifold relevant?

Note: The wedge product is especially helpful in differential forms because it allows the wedge product to govern the orientation (due to being an alternating tensor).

Additional tools used

- Wedge product: Anti-symmetric tensor
- Differential forms (which implement the wedge product)
- Doing calculus (i.e., doing derivatives and integrals) on differential forms

Proof of the Generalized Stokes' Theorem

The proof of the generalized Stokes' theorem is omitted for this presentation. Instead, an overview of how the generalized Stokes' theorem is a generalization of many other recognizable theorems in vector calculus is provided.

Green's Theorem

Restating the generalized Stokes' theorem as reference:

$$\int_{\partial A} \omega = \int_A d\omega.$$

Consider the $\mathbb{R}^2$ case where ω is a differential 1-form. Let $\omega = P\,dx + Q\,dy$ and D be a region where $C = \partial D$ is the boundary curve.

Interlude: A few computational rules

If a and b are differential 1-forms and f is a function:

$$d(a+b) = da + db$$

$$d(fa) = (df) \wedge a + f\,da$$

$$d(dx) = d(dy) = d(dz) = 0$$

$$df = f_x\,dx + f_y\,dy + f_z\,dz$$

Green's Theorem (cont.)

Calculating $d\omega$:

$$\begin{aligned}
d\omega = d(P\,dx + Q\,dy) &= d(P\,dx) + d(Q\,dy)\\
&= (dP) \wedge dx + P \cdot d(dx) + (dQ) \wedge dy + Q \cdot d(dy)\\
&= (P_x\,dx + P_y\,dy) \wedge dx + (Q_x\,dx + Q_y\,dy) \wedge dy\\
&= P_y\,dy \wedge dx + Q_x\,dx \wedge dy\\
&= (Q_x - P_y)\,dx \wedge dy
\end{aligned}$$

Green's Theorem (cont.)

We now plug the values defined and derived in the previous slides into the generalized Stokes' theorem,

$$\int_C P\,dx + Q\,dy = \int_D (Q_x - P_y)\,dx \wedge dy. \tag{3.1}$$

3.1 is also known as the **Green's Theorem**.

Remark

Green's Theorem is actually a specific case of the original Stokes' theorem.

Divergence Theorem

Consider a differential 2-form ω in $\mathbb{R}^3$. Let $\omega = P\,dy \wedge dz + Q\,dz \wedge dx + H\,dx \wedge dy$ and $G \subset \mathbb{R}^3$ be a domain bounded by a smooth surface S where $S = \partial G$ (G is the *closed solid* enclosed by S).

Similar to the previous example, we now proceed to calculating $d\omega$.

Divergence Theorem (cont.)

$$
\begin{aligned}
d\omega &= d(P\,dy \wedge dz + Q\,dz \wedge dx + H\,dx \wedge dy) \\
&= (dP)\,dy \wedge dz + (dQ)\,dz \wedge dx + (dH)\,dx \wedge dy \\
&= (P_x\,dx + P_y\,dy + P_z\,dz)\,dy \wedge dz \\
&\quad + (Q_x\,dx + Q_y\,dy + Q_z\,dz)\,dz \wedge dx \\
&\quad + (H_x\,dx + H_y\,dy + H_z\,dz)\,dx \wedge dy \\
&= P_x\,dx \wedge dy \wedge dz + Q_y\,dy \wedge dz \wedge dx + H_z\,dz \wedge dx \wedge dy \\
&= (P_x + Q_y + H_z)\,dx \wedge dy \wedge dz \\
&= \text{div } \mathbf{F}
\end{aligned}
$$

where $\mathbf{F} = \langle P, Q, H \rangle$.

Divergence Theorem (cont.)

Recall the generalized Stokes' theorem:

$$\int_{\partial A} \omega = \int_A d\omega.$$

Plugging values into the generalized Stokes' theorem:

$$\int_S P\,dy \wedge dz + Q\,dz \wedge dx + H\,dx \wedge dy = \int_G \text{div } \mathbf{F}, \tag{3.2}$$

a formula also known as the **Divergence Theorem**.

(Original) Stokes' Theorem

Assume ω to be a differential 1-form in $\mathbb{R}^3$. Let $\omega = P\,dx + Q\,dy + H\,dz$ and $S \subset \mathbb{R}^3$ be a surface with boundary curve C ($C = \partial S$).

Note that

$$d\omega = \text{curl } \mathbf{F}$$

where $\mathbf{F} = \langle P, Q, H \rangle$.

We will not prove this result here, the process is rather similar to the previous examples.

(Original) Stokes' Theorem (cont.)

We can now plug values into the generalized Stokes' theorem:

$$\int_C P\,dx + Q\,dy + H\,dz = \int_S \text{curl } \mathbf{F}. \tag{3.3}$$

3.3 is commonly called the (original) **Stokes' theorem**.

Fundamental Theorem of Calculus

We will now show how the fundamental theorem of calculus can be viewed as a specific case of the generalized Stokes' theorem.

Consider the case where ω is a 0-form in $\mathbb{R}^1$. Under these conditions, let ω be some function f and I be the interval $[a, b]$ whilst ∂I consists of the two endpoints, $\{a, b\}$. Assume the orientation of a to be negative and b to be positive.

Fundamental Theorem of Calculus (cont.)

Applying the generalized Stokes' theorem,

$$\int_{\{-a,b\}} f = \int_{[a,b]} df\,.$$

Expanding, we get:

$$f(b) - f(a) = \int_a^b f'(x)\,dx,$$

or the **fundamental theorem of calculus** (also called the Newton-Leibniz formula).

Conclusion

Thank you very much for listening to this presentation! Feel free to read my paper for additional details and proofs :).