THE GENERALIZED STOKES' THEOREM

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Abstract. This paper will go over the generalized Stokes' Theorem and provide a proof. We introduce the framework to understanding calculus on manifolds, and show how the original Stokes' Theorem, Divergence Theorem, Green's Theorem, and fundamental theorem of calculus can all be combined into the ultimate generalized Stokes' Theorem to which all of the theorems listed are special cases of.

Contents

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1. Introduction to Stokes' Theorem

The classical Stokes' Theorem is a theorem in vector calculus on $\mathbb{R}^3$ which, put briefly, states that the line integral of a vector field $\mathbf{F}$ over curve C, the boundary of surface A, is equal to the flux of curl $\mathbf{F}$ through A. Despite bearing the name of George Stokes, the original Stokes' Theorem was first stated in a letter from Lord Kelvin to George Stokes. It was then vastly generalized in 1945 by Élie Cartan into its modern form, the generalized Stokes' Theorem, a result that spans several theorems of vector calculus, including Isaac Newton's fundamental theorem of calculus. The generalized Stokes' Theorem essentially generalizes the fundamental theorem of calculus to 2-dimensional line integrals and 3-dimensional surface integrals, such that the classical theorems ie. Divergence Theorem, Green's Theorem, original Stokes' Theorem all become special cases beneath the overall umbrella of the generalized Stokes' Theorem.

2. General definitions

This section will provide the conceptual definitions needed to understand later theorems. Feel free to skip this section and refer to it later when particular concepts are confusing.

2.1. **Differentials.**

Definition 2.1. Let $f : U \to \mathbb{R}$ be a function on a domain $U \subset V$ in a vector space V. The function f is called *differentiable* at a point $x \in U$ if there exists a linear function $l : V_x \to \mathbb{R}$ such that

$$f(x+h) - f(x) = l(h) + o(||h||)$$

for any sufficiently small vector h, where $o(t)$ is any function such that $\underset{\frac{o(t)}{t}\to 0}{t \to 0}$.

In more recognizable terms (ie. similar to how derivatives are defined),

Definition 2.2. f is differentiable at $x \in U$ if for any $h \in V_x$ there exists a limit

$$l(h) = \lim_{t\to 0} \frac{f(x+th) - f(x)}{t}$$

and the limit l(h) *linearly* depends on h.

(See later in this subsection if the linear dependency on h part does not quite make sense, we discuss it in detail when covering how to conceptually understand differentials).

So far, differentials probably seem rather similar to derivatives: which, in fact, is true. We can define differentials in terms of derivatives,

$$l(h) = d_x f(h)$$

is called the *directional derivative* of f at point x in the direction of h. Recalling the definition of directional derivatives from a calculus textbook [Ant16], we see that differentials are probably quite familiar even to individuals who have not encountered the exact term "differential" before.

Definition 2.3 (Directional derivative)**.** If $f(x, y)$ is differentiable at (x_0, y_0), and if $u = \langle u_1, u_2 \rangle$ is a unit vector, then the directional derivative $D_u f(x_0, y_0)$ exists and is given by

$$D_u f(x_0, y_0) = f_x(x_0, y_0)u_1 + f_y(x_0, y_0)u_2$$

The definition in 3-space directly follows from the definition in 2-space (except with one additional variable).

Examining the equation, we see that $f_x(x_0, y_0)u_1 + f_y(x_0, y_0)u_2$ is essentially a differential equation: $u_1 dx + u_2 dy$.

It may appear pointless to have "differentials" if they are, by definition, derivatives: note, however, that derivatives and differentials conceptually measure different things. Derivatives measure the rate of change, while differentials measure the change itself. For example, the rate of inflation is different from the actual amount of inflation. This idea of differentials as measuring the change itself is what we often use in integrals: the dx is, in essence, a small, calculus-sized change in the x direction.

By now, hopefully differentials seem much more familiar than before. Let us add in another definition that should be familiar from previous calculus classes:

Definition 2.4 (Differentiability)**.** If partial derivatives exist in a neighborhood of point a and are continuous at point a then f is differentiable at point a. The function f is called differentiable on the whole domain U if it is differentiable at each point of U.

Returning to the idea of the limit $l(h)$ (or the directional derivative) being linearly dependent on h, let us conceptualize what *differentiability* indicates physically. The differentiability of a function means that at a small (ie. calculus sized) scale near a point x the function behaves approximately like a linear function, or the differential of the function at point x. This idea is essentially a restatement of the concept behind linear approximation, that the tangent line becomes a good approximation of the curve when we zoom in enough. However, this linear function (the differential of f at x) varies from point to point, so we call the family $\{d_x f\}_{x \in U}$ of all these linear functions the *differential* of f, and denote it df.

2.2. **Smooth functions.** Let us begin with the usual, common definition of smoothness: a function is commonly taught as "smooth" if its partial derivatives are continuous. In more formal terms, a C^1 smooth function is a function with continuous first partial derivatives. In terms of differentials, f is C^1 smooth if the differential $d_x f$ continuously depends on the point $x \in U$.

More generally, we arrive at the definition:

Definition 2.5. For $k \geq 1$, a function $f : U \to \mathbb{R}$ is called C^k smooth if all its partial derivatives up to order k are continuous in U.

3. Defining Manifolds

3.1. **Smooth manifolds.** Definition 2.5 holds only when its domain is an open set U: to adapt the concept of smoothness to more general spaces, we give the following definition from [GP10].

Definition 3.1. A map $f : X \to \mathbb{R}^m$ defined on an arbitrary subset X in $\mathbb{R}^n$ is called *smooth* if it may be locally extended to a smooth map on open sets.

From this definition, smoothness is a local property: $f : X \to \mathbb{R}^m$ is smooth if it is smooth in a neighborhood of each point of X.

3.2. **Manifolds.** A manifold is a topological space that is *locally* Euclidean; i.e., the neighborhood around every point is topologically the same as the open unit ball in $\mathbb{R}^n$. To better conceptualize this idea of a manifold, consider the Earth as a manifold. The controversy over the shape of the Earth, particularly the flat-earth theory which argues that the Earth is flat versus the modern evidence that it is round, arises essentially from the fact that on the small scales that we see, the Earth appears flat. Humans do not experience the roundness of the Earth because we reside in tiny, zoomed in parts of the globe that are "flat." In general, any object that is nearly "flat" on small scales is a manifold, so manifolds are in essence a generalization of all the objects we could live on in which we would face the flat-round Earth problem. Figure 3.1 gives a visual representation of this local linearity (flatness).

Manifolds are one of the most important classes of spaces in mathematics, spanning differential geometry, theoretical physics, and algebraic topology. In this paper, we will limit ourselves to manifolds that are submanifolds (Definition 3.7) of Euclidean space $\mathbb{R}^n$.

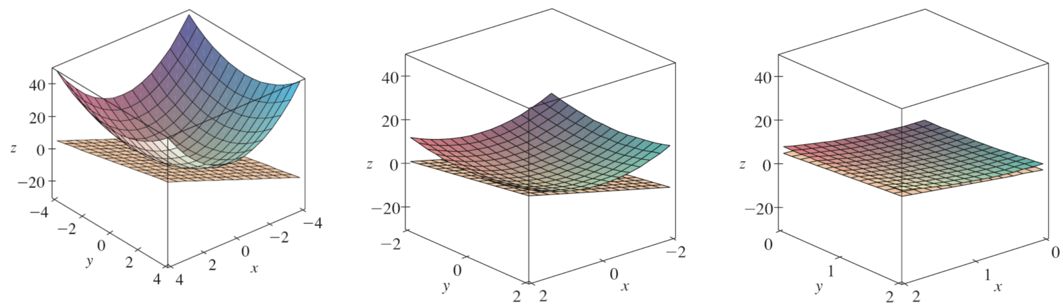

Figure 3.1. Local linearity

A note on terminology: when we refer to "manifold" in this paper, we are referring to a manifold with boundary. The two are not generally equivalent.

Let us now dive into the formal definitions of manifolds. Further details regarding these definitions can be found at [LL03], [Mun91], and [Eli18].

Definition 3.2 (Defining manifolds)**.** M is a topological manifold of dimension k (topological k-manifold) if it has the following properties:

(1) M is a *Hausdorff* space: for every pair of distinct points $p, q \in M$, there are disjoint open subsets $U, V \subseteq M$ such that $p \in U$ and $q \in V$.
(2) M is a *second-countable* space: there exists a countable basis for the topology of M.
(3) M is *locally Euclidean* of dimension n: each point of M has a neighborhood that is homeomorphic to an open subset of $\mathbb{R}^n$.

Note: (1) essentially states that a Hausdorff space is a topological space where for any 2 distinct points there exist neighborhoods of each which are disjoint[1] from each other.

Let us now define a homeomorphism, also known as a continuous transformation.

Definition 3.3 (Homeomorphism)**.** A map $f : U \to U'$ is called a homeomorphism if it is a continuous one-to-one map (i.e., it is bijective) which has a continuous inverse $f^{-1} : U' \to U$.

A homeomorphism is essentially an equivalence relation and one-to-one correspondence between points in two geometric figures or topological spaces. Diffeomorphisms are a type of homeomorphism that will be of more interest for us in this paper.

[1]Disjoint set have no element in common.

Definition 3.4 (Diffeomorphism)**.** A map $f : U \to U'$ is called a C^k-diffeomorpism, $k = 1, \dots, \infty$, if it is a C^k-smooth, one-to-one map which has a C^k-smooth inverse $f^{-1} : U' \to U$.

A diffeomorphism is an isomorphism of smooth manifolds, an invertible function (ex. f) that maps one differentiable manifold X to another (Y) such that both f and f^{-1} are differentiable. X and Y are *diffeomorphic* if such a map exists and are, for our purposes, intrinsically equivalent.

Let us now formally define the boundary of manifolds.

We state the following theorem without proof[2].

Theorem 3.5. *Let M be a k-manifold in $\mathbb{R}^n$, of class C^r. Let $\alpha_0 : U_0 \to V_0$ and $\alpha_1 : U_1 \to V_1$ be coordinate patches on M, with $W = V_0 \cap V_1$ non-empty. Let $W_i = \alpha_i^{-1}(W)$. Then the map*

$$\alpha_i^{-1} \circ \alpha_0 : W_0 \to W_1$$

is of class C^r, and its derivative is non-singular.

The boundary of a manifold can then be defined as below.

Definition 3.6. Let M be a k-manifold in $\mathbb{R}^n$; let $\mathbf{p} \in M$. If there is a coordinate patch $\alpha : U \to V$ on M about $\mathbf{p}$ such that U is open in $\mathbb{R}^k$, we say $\mathbf{p}$ is an **interior point** of M. Otherwise, we say $\mathbf{p}$ is a **boundary point** of M. We denote the set of boundary points of M by ∂M, and call this set the **boundary** of M.

Figure 3.2 gives a visual representation of an interior point versus a boundary point on manifold M.

After defining manifolds, we define a submanifold as below:

Definition 3.7 (Submanifolds)**.** Let V be an n-dimensional vector space. A subset $A \subset V$ is called a k-dimensional submanifold of V, or simply a k-submanifold of V, $0 \leq k \leq n$, if for any points $a \in A$ there exists a local coordinate chart $(U_a, u = (u_1, \dots, u_n) \to \mathbb{R}^n)$ such that $u(a) = 0$ (i.e. the point a is the origin in this coordinate system) and

$$A \cap U_a = \{u = (u_1, \dots, u_n) \in U_a; u_{k+1} = \cdots = u_n = 0\}.$$

We can also define a submanifold with boundary (which, recall, is *not* equivalent a submanifold):

[2]See [Mun91] for a proof.

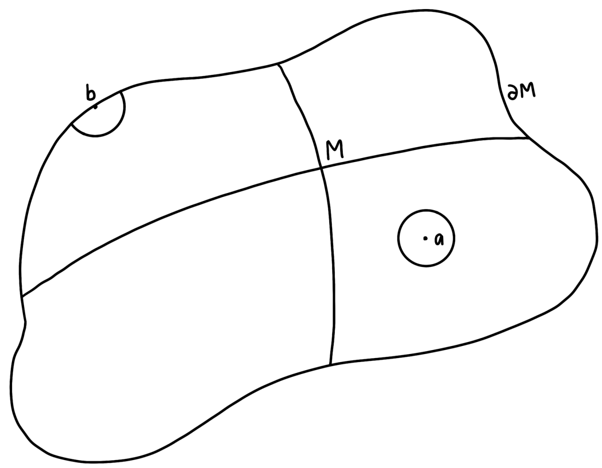

Figure 3.2. Manifold M with interior point a and boundary point b

Definition 3.8 (Submanifold with boundary)**.** A subset $A \subset V$ is called a k-dimensional submanifold of V, or simply a k-submanifold of V, $0 \leq k \leq n$, if for any points $a \in A$ there is a neighborhood $U_a \ni a$ in V and local (curvi-linear) coordinates $(u_1, \ldots, u_n)$ in U_a with the origin at a if one of two conditions is satisfied: the condition in Definition 3.7 or the following condition

$$A \cap U_a = \{u = (u_1, \ldots, u_n) \in U_a; u_1 \geq 0, u_{k+1} = \cdots = u_n = 0\}.$$

3.3. **Orientation.** The orientation of a manifold is likely not a new concept: essentially, defining orientation is like defining the "direction" by which we approach the manifold. Orientation is often introduced as "positive" or "negative" in calculus textbooks, but we may note that this so-called positiveness or negativeness is a question of convention. For instance, "positive" orientation is often defined as counter-clockwise orientation of the plane, but "counterclockwise" is dependent on which side we look at the plane. Thus, we must note that orientation, as we usually know it, is a physical rather than mathematical notion. We now give a mathematical definition of orientation.

Definition 3.9. Two bases $v_1, \ldots, v_k$ and $w_1, \ldots, w_k$ of a vector space V define the same orientation of V if the matrix of the transformation from one base to the other has a positive determinant.

Remark 3.10. From the above definition, we can derive that: if there are 3 bases, and the 1st and the 2nd define the same orientation, and the 2nd and the 3rd define the same orientation, then the 1st and the 3rd define the same orientation. In other words, the law of transitivity but for orientation.

Note that the geometric idea behind this definition of orientation makes inherent sense. The determinant D of a matrix transforming one base of V to the other (ie. matrix transforming the "coordinates") essentially gives the "area proportion constant" between the two bases of V. For example, in 2D, when we change from the basis vectors i and j to another two basis vectors, the unit square that i and j formed morphs into another unit (a rectangle, parallelogram, etc. depending on the transformation). The consequent change in area of this unit square is the determinant. So if the change of basis vectors caused the new unit shape to be $2\times$ the area of the i and j unit square, then the determinant of the transformation matrix equals 2. A negative determinant thus indicates a "flipping" of orientation from the original unit square. Therefore, if two bases of V have a transformation matrix that yields a positive determinant, no "flipping" from the original orientation has occurred (indicating that they are of the same orientation, as the mapping from one to the other did not require flipping).

Now, let us define orientation for manifolds. Firstly, a remark to read (even if it is the only thing you read beyond this point):

Remark 3.11. A bit of thought on why orientation of manifolds matters before we go through *all* those definitions and think, for what *reason* are we going through this in the first place?
First of all, the point, or at least our point, of formally defining just manifolds in general (not even their orientation) is to be able to integrate over these surfaces. Below is an integral of the type we hope to solve.

$$\int_M \omega = \int_{\text{Int } U} \alpha^* \omega$$

However, if we have only defined manifolds in general (re: the definitions in the subsections above), we run into an issue: this integral is invariant under reparametrization *only up to sign*. In other words, reparametrizing the manifold M may cause the solution of the integral to change sign. To have the integral be *well-defined*, we need an extra condition on M: **orientation**.

Now that we have a basic understanding of why defining the orientation of manifolds is important, let us dive into the actual definitions of orientation and orientability.

We begin by formally defining orientability for manifolds [Mun91].

Definition 3.12 (Orientability, formally defined)**.** Let M be a k-manifold in $\mathbb{R}^n$. Given coordinate patches $\alpha_i : U_i \to V_i$ on M for $i = 0, 1$: they **overlap positively** if the transition function $\alpha_1^{-1} \circ \alpha_0$ is orientation-preserving. If M can be covered by a collection of coordinate patches each pair of which overlap positively (if they overlap at all), then M is said to be **orientable**. Otherwise, M is said to be **non-orientable**.

Before we delve into how to obtain orientation of manifolds, let us redefine **orientation** in more formal terms.

Definition 3.13 (Orientation)**.** Let M be a k-manifold in $\mathbb{R}^n$. Suppose M is orientable. Given a collection of coordinate patches covering M that overlap positively, let us adjoin to this collection all other coordinate patches on M that overlap these patches positively. It is easy to see that the patches in this expanded collection overlap one another positively. This expanded collection is called an **orientation** on M. A manifold M together with an orientation of M is an **oriented manifold**.

To define the actual orientation of manifolds, we start with defining *vector bundles*.

Definition 3.14 (Vector bundle)**.** A vector bundle of rank r over a set $A \subset V$ is a family of r-dimensional vector subspaces $L_a \subset V_a$, parameterized by points of A and continuously (C^m-smoothly) depending on a. Put precisely, each point $a \in A$ has a neighborhood $U \subset A$ such that there exists linear independent vector fields $v_1(a), \ldots, v_r(a) \in L_a$ which continuously depend on a.

In different terms, a vector bundle is a topological construction where we have a family of vector spaces parameterized by another space U (U can be a topological space, a manifold, etc.): to every point u of the space U, we associate a vector space V_u such that these vector spaces fit together to form another space of the same kind as U, called a vector bundle over U.

We now define the *tangent bundle* of a submanifold, which is a more general notion of a vector bundle.

Definition 3.15 (Tangent bundle)**.** A collection of all tangent spaces $\{T_aA\}_{a \in A}$ to a submanifold A is called its tangent bundle and denoted by TA or $T(A)$.

Another important vector bundle over a submanifold A is its normal bundle $NA = N(A)$.

Definition 3.16 (Normal bundle)**.** A vector bundle of rank $n - k$ formed by orthogonal complements $N_aA = T_a^{\perp}A \subset V_a$ of the tangent spaces T_aA of A.

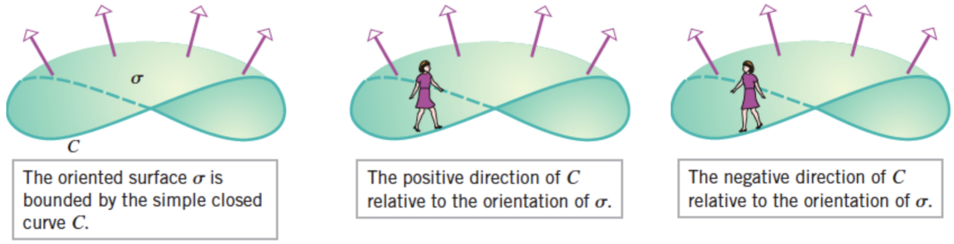

Figure 3.3. Orientation of 2-manifold in $\mathbb{R}^3$ from [Ant16]. Note how the direction the head of the person (the normal vector of the manifold) defines the orientation of the surface.

We assume here that V is Euclidean space.

We are now ready to define the orientation and co-orientation of a submanifold k.

Definition 3.17 (Orientation of vector bundles)**.** An orientation of a submanifold k is the same as an orientation of its tangent bundle $T(A)$. A co-orientation of a k-submanifold A is an orientation of its normal bundle $N(A) = T \perp A$ in V .

Note that not all bundles are orientable: some bundles have no orientation. However, if L is orientable and A is connected, then L has exactly *two* orientations.

Connecting the orientation of vector bundles to the orientation of manifolds (surfaces):

Definition 3.18 (Orientation of surfaces, formally defined)**.** A differentiable manifold M is orientable if and only if its tangent bundle is orientable. The orientation of the surface is induced by its co-orientation by the normal vector n (or, for the entire surface, the unit *normal* vector field of M). The orientation of the boundary is induced by the orientation of the surface.

The idea of essentially the normal vectors defining the orientation can be seen in simpler examples, such as the definition of orientation of a 2-manifold in $\mathbb{R}^3$ in Figure 3.3 from a calculus textbook.

See Figure 3.4 for a visual idea of Definition 3.18.

Figures 3.5 and 3.6 give examples of non-orientable surfaces.

Before we close up this section, let us zoom out to review the ultimate point of this section: to provide a formal definition of the surfaces which the later theorems in this paper will integrate over. Hopefully, the general concepts covered were not completely foreign (orientation, boundary of surfaces, tangent

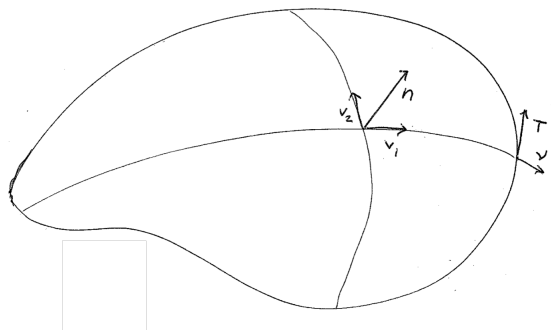

Figure 3.4. Orientation of the surface [Eli18]

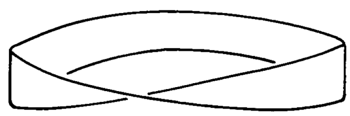

Figure 3.5. Möbius strip, a 2-manifold in $\mathbb{R}^3$. Note how the manifold has no continuous unit vector field. [Mun91]

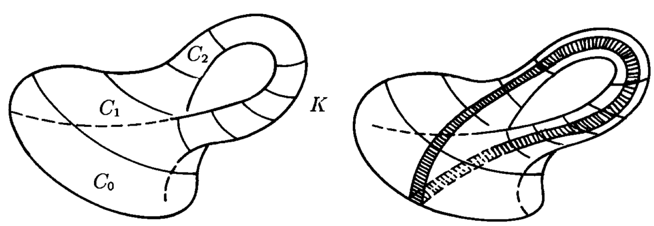

Figure 3.6. Klein bottle, a 2-manifold in $\mathbb{R}^3$. Note how the manifold, like the Möbius strip, has no continuous unit vector field. [Mun91]

and normal vectors etc.), such that the formalization of certain concepts was the more foreign aspect.

4. Differential forms

4.1. **Tensors: the Wedge Product.**

Tensors. Let us begin by defining the tensor product: given a k-linear function ϕ and a l-linear function ψ, the tensor product of the functions ϕ and ψ $\phi \otimes \psi$ is a $(k + l)$-linear function. By definition,

$$\phi \otimes \psi(X_1, \ldots, X_k, X_{k+1}, \ldots, X_{k+l}) := \phi(X_1, \ldots, X_k) \cdot \psi(X_{k+1}, \ldots, X_{k+l}). \tag{4.1}$$

Applying this definition, the tensor product of two linear functions l_1 and l_2 equals the bilinear function (ex. $f(x, y) = xy$ is bilinear in that it combines linear x and y) $l_1 \otimes l_2$. Assuming U and V to be the inputs to l_1 and l_2 respectively,

$$l_1 \otimes l_2(U, V) = l_1(U) \cdot l_2(V). \tag{4.2}$$

Symmetric and anti-symmetric tensors. We begin with the definitions.

Definition 4.1 (Symmetric tensor)**.** A tensor is **symmetric** if it remains unchanged under the transposition of any two of its arguments:

$$f(X_1, \ldots, X_i, \ldots, X_j, \ldots, X_k) = f(X_1, \ldots, X_j, \ldots, X_i, \ldots, X_k).$$

The above condition can also be written as

$$f(X_{i_1}, \ldots, X_{i_k}) = f(X_1, \ldots, X_k)$$

for any permutation $i_1, \ldots, i_k$ of indices $1, ..., k$.

Definition 4.2 (Anti-symmetric tensor)**.** A tensor is **anti-symmetric** if it changes its sign under the transposition of any two of its arguments:

$$f(X_1, \ldots, X_i, \ldots, X_j, \ldots, X_k) = -f(X_1, \ldots, X_j, \ldots, X_i, \ldots, X_k).$$

The above condition can also be written as

$$f(X_{i_1}, \ldots, X_{i_k}) = (-1)^{\mathrm{inv}(i_1 \ldots i_k)} f(X_1, \ldots, X_k)$$

for any permutation $i_1, \ldots, i_k$ of indices $1, ..., k$, where $\mathrm{inv}(i_1 \ldots i_k)$ is the number of inversions in the permutation $i_1, \ldots, i_k$.

Anti-symmetric tensors are of more interest to us in this paper, for reasons that will become clear when we define the wedge product. Before we do that, however, let us review how to create symmetric or anti-symmetric tensors from arbitrary tensors.

Let $f(X_1, \ldots, X_k)$ be a k-tensor. Set ...

Definition 4.3 (Constructing symmetric tensors)**.**

$$f^{\mathrm{sym}}(X_1, \ldots, X_k) := \sum_{(i_1 \ldots i_k)} f(X_{i_1}, \ldots, X_{i_k})$$

Definition 4.4 (Constructing anti-symmetric tensors)**.**

$$f^{\text{asym}}(X_1,\dots,X_k) \coloneqq \sum_{(i_1\dots i_k)} (-1)^{\text{inv}(i_1\dots i_k)} f(X_{i_1},\dots,X_{i_k})$$

Remark 4.5. Hint: note how the construction formulas for symmetric and anti-symmetric tensors relate to the definitions if they are confusing.

Wedge product. We now define the wedge product[3] [Eli18].

Remark 4.6. "Exterior k-forms" is another name for skew-symmetric k-linear functions

Definition 4.7 (Wedge product)**.** Let ϕ be an exterior k-form and ψ an exterior l-form. The exterior $(k+l)$-form $\psi \wedge \psi$, the wedge product of ϕ and ψ, is defined as

$$\phi \wedge \psi \coloneqq \frac{1}{k!l!}(\phi \otimes \psi)^{\text{asym}}.$$

Expanded,

$$\begin{aligned} &\phi \wedge \psi(X_1,\dots,X_k,X_{k+1},\dots,X_{k+l}) \\ &= \frac{1}{k!l!} \sum_{i_1\dots i_{k+l}} (-1)^{\text{inv}(i_1\dots i_{k+l})}\, \phi(X_1,\dots,X_k)\, \psi(X_{k+1},\dots,X_{k+l}). \end{aligned}$$

where the sum is taken over all permutations of indices $1,\dots,k+l$.

Let us now briefly go over a few properties of the wedge product as an operation.

Remark 4.8. For any exterior k-form ϕ and exterior l form ψ, the wedge product has the following basic properties:

- $\phi \wedge \psi = (-1)^{kl}\psi \wedge \phi$
- $(\phi_1 + \phi_2) \wedge \psi = \phi_1 \wedge \psi + \phi_2 \wedge \psi$
- $(\lambda\phi) \wedge \psi = \lambda(\phi \wedge \psi)$
- Associativity: $(\phi \wedge \psi) \wedge \omega = \phi \wedge (\psi \wedge \omega)$

We will not prove these properties in this paper, see [Eli18] or [GP10] for proofs of the above properties.

Understanding the wedge product. Now that we have covered the definition of the wedge product, let us take a look at what the wedge product means conceptually. We begin this discussion of the wedge product's conceptual meaning with a few more theorems regarding its properties.

Remark 4.9. Two more properties of the wedge product:

[3]Also known as the exterior product.

(1) $u \wedge v = -v \wedge u$
(2) $u \wedge u = 0$

Theorem 4.10. *The product $u \wedge v$ determines the area of the parallelogram spanned by u and v, as well as the plane containing these vectors when there is a unique such plane.*

Do these properties of the wedge product appear familiar? Recall the properties listed below of the *cross product* in $\mathbb{R}^3$:

Theorem 4.11 (Properties of the cross product)**.** *If u and v are any vectors in 3 space,*

$$u \times v = -v \times u$$
$$u \times u = 0$$

Compare Remark 4.9 with Theorem 4.11.

Theorem 4.12 (Area of a parallelogram from 2 vectors)**.** *The area A of the parallelogram with u and v as adjacent sides is*

$$A = ||u \times v||$$

Remark 4.13. Cross product also defines a plane when given two vectors u and v on this plane (given that they are not parallel): $u \times v$ defines the normal vector to the plane, and $\mathbf{n}$ & a point (provided by either vector) define a plane. If $\mathbf{n} = u \times v = \langle A, B, C\rangle$ and the point on the plane $= \langle u_1, u_2, u_3\rangle$,

$$\text{Plane in } \mathbb{R}^3\text{: } A(x - u_1) + B(y - u_2) + C(z - u_3) = 0.$$

Compare Theorem 4.10 with Theorem 4.12 and Remark 4.13.

Thus, we see that the wedge product is in a sense a parallel of the cross product in higher dimensions. The similarity between the wedge product and the cross product can be further seen through a comparison of their formulas in $\mathbb{R}^3$.

Wedge product in $\mathbb{R}^3$:

$$\langle a, b, c\rangle \wedge \langle d, e, f\rangle = \frac{1}{2}\begin{bmatrix} 0 & ae - bd & af - cd \\ bd - ae & 0 & bf - ce \\ cd - af & ce - bf & 0 \end{bmatrix}$$

Note the similarities of the wedge product with the cross product in $\mathbb{R}^3$:

$$\langle a, b, c\rangle \times \langle d, e, f\rangle = \langle bf - ce, cd - af, ae - bd\rangle$$

Remark 4.14. Whilst the majority of this section focuses on the similarities between the cross product and the wedge product in $\mathbb{R}^3$ so as to give the wedge product more conceptual meaning, they do share fundamental differences: namely, the wedge product produces a matrix and the cross product produces a vector. With this note, we continue our discussion of the wedge product's cross product like nature in terms of physical meaning.

Here, we will use differential forms, which are defined in the next subsection (so feel free to jump there and then jump back). However, the idea of the wedge product should be conveyed through the next few lines even without much background in differential forms. Differential forms and their integration over a surface is the ultimate reason this paper covers the wedge product, so we will end this section on the wedge product with an application of it in differential forms.

Firstly, a bit of context on differential forms (refer to 4.2 for details). A 1-form is an expression that can be integrated over a curve, ex. $f\,dx$; a 2-form is an expression that can be integrated over a surface, ex. $g\,dx \wedge dy$.
We can convert a vector field $\mathbf{F} = f\mathbf{i} + g\mathbf{j} + h\mathbf{k}$ into a 1-form:

$$f\mathbf{i} + g\mathbf{j} + h\mathbf{k} \Rightarrow f\,dx + g\,dy + h\,dz \tag{4.3}$$

We can also convert $\mathbf{F}$ into a 2-form:

$$f\mathbf{i} + g\mathbf{j} + h\mathbf{k} \Rightarrow f\,dy \wedge dz + g\,dx \wedge dz + h\,dx \wedge dy \tag{4.4}$$

In the 2-form above, we convert $\mathbf{i}$ into $dy \wedge dz$, $\mathbf{j}$ into $dx \wedge dz$, and $\mathbf{k}$ into $dx \wedge dy$. Here, $\wedge$ essentially acts as the cross product: $\mathbf{i}$ = the cross product of dy and dz. Similarly, to go from a 2-form to a vector, we can replace dx, dy, and dz with $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ and then take the cross products of $\mathbf{i}\ \mathbf{j}\ \mathbf{k}$.

Remark 4.15. Hopefully, the above discussion offered a somewhat better conceptual understanding of the wedge product. We will now progress to our main application of the wedge product: differential forms.

4.2. **Defining differential forms.** Although the term *differential forms* may sound unfamiliar, any introductory calculus class has likely used differential forms before, especially the integration of them. Let us begin our discussion of differential forms with an example that is probably rather recognizable.

Say we have a 1D manifold [a, b] with "positive" orientation, i.e., $a < b$. Then

$$\int_a^b f(x)\,dx$$

is the integral of the differential 1-form $f(x)\,dx$ over [a, b].

Another example of a differential form has already appeared in this paper: vector fields can be rewritten as differential 1-forms and differential 2-forms, as seen in Equation 4.3 and 4.4.

Now that we have a general understanding of what differential forms are, we state the following formal definition from [GP10]

Definition 4.16 (k-forms, formally defined)**.** Let M be a smooth manifold with or without boundary. A k-form on M is a function ω that assigns to each point $m \in M$ an alternating k-tensor $\omega(x)$ on the tangent space of M and m; $\omega(x) \in \Lambda^k[T_m(M)^*]$.

We can list a few more differential forms:

$$\sum_i f_i \, dx_i$$

$$\sum_{i<j} f_{ij} \, dx_i \wedge dx_j$$

$$\sum_{i<j<k} f_{ijk} \, dx_i \wedge dx_j \wedge dx_k$$

Following this pattern, any differential k-form can be expressed as:

$$\sum_{1 \leq i_i < i_2 \cdots < i_k \leq n} f_{i_1 \ldots i_k} \, dx_{i_1} \wedge \cdots \wedge dx_{i_k}$$

Before we move on, let us note why differential forms are important and relevant to this paper. Firstly, we zoom out from this subsection and connect differential forms with exterior forms and the wedge product.
For our purposes, we will simply define the exterior form in an informal fashion. See [Eli18] for greater detail regarding them.

Definition 4.17 (Exterior form, informally defined)**.** The exterior form is the result of exterior products (i.e., wedge products).

Now, we provide another definition of differential forms in terms of exterior forms.

Definition 4.18 (Differential forms, defined in terms of exterior forms)**.** Differential k-forms are *fields* of exterior k-forms (similar to the relationship between vectors and vector *fields*).

After clarifying some terminology, let us dive into why differential forms are relevant. Essentially, differential forms allow us to approach topics in calculus in a way that is *separate from coordinates.* Differential forms, in essence, measure some kind of infinitesimally small length, area, volume, . . . and so on in the kth dimension. A differential 0-form is simply an arbitrary real-valued function on M.

Differential forms use the wedge product to keep track of the length, area, volume, etc. while preserving the orientation of the manifold. Note that when we integrate a differential 1-form over an interval $[a, b]$, if we switch the order of a and b (i.e., integrate over $[b, a]$ instead), the result of the two integrals will be equal in magnitude but with opposing sign. Written in equation form,

$$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx \tag{4.5}$$

Recall that the wedge product is essentially an alternating (anti-symmetric) tensor: this aspect of it allows the wedge product to govern the orientation and order in which we take some integral. In other words, it preserves the notion of the negation we see in Equation 4.5 after reversing orientation. Thus, the wedge product can both track the length/area/volume etc. we measure whilst preserving the orientation, allowing us to have a unified sense of what we are measuring and keeping orientation in order. Instead of having to specify $[a, b]$ or $[b, a]$, the wedge product allows us to have our integral such that when we integrate over the same manifold but in a different orientation, our integral shows as measuring the same length/area/volume etc. while preserving orientation.

Pulling back to differential forms: the ultimate idea is that differential forms are able to work separate from the coordinate system due to the wedge product. This attribute of differential forms allows them to be globally defined on manifolds, and the global definition of integrands makes possible global integration.

Example. We will now work on an example which uses differential forms and should appear familiar (or at least be insightful, especially in our penultimate section).
We begin with a differential 1-form: $\omega = F(x, y, z)\,dx + G(x, y, z)\,dy + H(x, y, z)\,dz$. Note that ω is essentially the vector field $\mathbf{V} = \langle F, G, H\rangle$. We want to define its *exterior derivative* (see next subsection for details) $d\omega$ which will be a differential 2-form (k+1). Recall the following rules, where a and b are differential 1-forms and f is a function:

$$d(a + b) = da + db$$
$$d(fa) = (df) \wedge a + f da$$
$$d(dx) = d(dy) = d(dz) = 0$$

$$df = f_x\,dx + f_y\,dy + f_z\,dz \text{ where } f_x = \frac{\partial f}{\partial x} \text{ etc.}$$

$$\begin{aligned}
d\omega &= d(F\,dx + G\,dy + H\,dz) = d(F\,dx) + d(G\,dy) + d(H\,dz)\\
&= (dF) \wedge dx + Fd(dx) + (dG) \wedge dy + Gd(dy) + (dH) \wedge dz + Hd(dz)\\
&= (F_x\,dx + F_y\,dy + F_z\,dz) \wedge dx\\
&\quad + (G_x\,dx + G_y\,dy + G_z\,dz) \wedge dy\\
&\quad + (H_x\,dx + H_y\,dy + H_z\,dz) \wedge dz\\
&= F_y\,dy \wedge dx + F_z\,dz \wedge dx + G_x\,dx \wedge dy + G_z\,dz \wedge dy +\\
&\quad + H_x\,dx \wedge dz + H_y\,dy \wedge dz\\
&= (G_x - F_y)\,dx \wedge dy + (H_y - G_z)\,dy \wedge dz + (F_z - H_x)\,dz \wedge dx\\
&= \boldsymbol{\nabla} \times \mathbf{V} = \text{curl } \mathbf{V} \qquad (4.5)
\end{aligned}$$

Remark 4.19. We will continue to build on this discussion of differential forms in the next sections, where we approach taking the derivative and integral of differential k-forms.

5. Derivatives of differential forms: Exterior derivatives

The *exterior derivative* is essentially a derivative defined for deriving of differential forms. We have already encountered this operation in this paper (see Example 4.2). The exterior derivative in the current form was introduced by Élie Cartan, and transforms a differential k-form is a $(k+1)$-form. We begin this section with an example of the exterior derivative operator d turning a differential 0-form into a 1-form.

Example. Let f be the differential 0-form (i.e., function) on $\mathbb{R}^n$ $f(x_1, \ldots, x_n)$. Now, let us do a bit of brainstorming to find a d operator which would make it such that:

(1) df is a differential 1-form
(2) To evaluate this 1-form, we would put in a point $p \in \mathbb{R}^n$ and a vector $v \in T_p\mathbb{R}^n$ (in accordance with the general evaluation of differential forms)

The *directional derivative* seems like a likely candidate, in that it both takes derivative and needs the input of a point and a vector (see 2.3). Expanding on this idea,

$$(df)_p(v) = D_v f\Big|_{x=p} = \nabla f\Big|_p \cdot v$$
$$= \frac{\partial f}{\partial x_1} v_1 + \cdots + \frac{\partial f}{\partial x_n} v_n = \frac{\partial f}{\partial x_1} dx_1(v) + \cdots + \frac{\partial f}{\partial x_n} dx_n(v).$$

We can then use the differential 0-form to 1-form case to generalize to differential k-forms: setting $dx_I = dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_k}$,

$$d(f\, dx_I) = \frac{\partial f}{\partial x_1} dx_1 \wedge dx_I + \cdots + \frac{\partial f}{\partial x_n} dx_n \wedge dx_I.$$

The results of Example 5 should reflect into the following formal definition of the exterior differential and its properties.

Definition 5.1. (Exterior derivative) If $\omega = \sum_{i_1 < \cdots < i_k} a_{i_1 \ldots i_k}\, dx_{i_1} \wedge \cdots \wedge dx_{i_k}$ is a smooth differential k-form on an open subset of $\mathbb{R}^n$ where $a_{i_1 \ldots i_k}$ are functions, define:

$$d\omega \coloneqq \sum_{i_1 < \cdots < i_k} da_{i_1 \ldots i_k} \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_k}.$$

Note that this definition is independent of the choice of coordinate system; we will omit the proof in this paper but see [Eli18] for details.

We now list the most important properties of this definition of the exterior derivative:

Theorem 5.2. *Define θ as a function and ω, ω_1, and ω_2 as differential k-forms. The exterior differentiation operator d, defined on smooth forms on the open $U \subset \mathbb{R}^n$, possesses the following three properties:*

(1) $d(\omega_1 + \omega_2) = d\omega_1 + d\omega_2$
(2) $d(\omega \wedge \theta) = (d\omega) \wedge \theta + (-1)^k \omega \wedge d\theta$
(3) $d(d\omega) = 0$

To give an example of applying the exterior derivative operator d before we close out this section, let us calculate completely the operator d in $\mathbb{R}^3$.

Differential 0-forms → 1-forms. If f is a function on $\mathbb{R}^3$, then

$$df = f_x\, dx_1 + f_y\, dx_2 + f_z\, dx_3$$

where $\langle f_x, f_y, f_z \rangle = \nabla f$, the *gradient* vector field of f.

Differential 1-forms → 2-forms. This is the same computation process as Example 4.2, so we will skip the process and directly state the result. If

$$\omega = f_1\, dx_1 + f_2\, dx_2 + f_3\, dx_3,$$

then

$$d\omega = g_1\, dx_2 \wedge dx_3 + g_2\, dx_3 \wedge dx_1 + g_3\, dx_1 \wedge dx_2,$$

where

$$g_1 = \frac{\partial f_3}{\partial f_2} - \frac{\partial f_2}{\partial f_3}, \quad g_2 = \frac{\partial f_1}{\partial f_3} - \frac{\partial f_3}{\partial f_1}, \quad g_3 = \frac{\partial f_2}{\partial f_1} - \frac{\partial f_1}{\partial f_2}.$$

Defining $\mathbf{F}$ and $\mathbf{G}$ to be the vector fields $\langle f_1, f_2, f_3 \rangle$ and $\langle g_1, g_2, g_3 \rangle$, $\mathbf{G} = \text{curl } \mathbf{F}$.

Differential 2-forms → 3-forms. Let

$$\omega = f_1\, dx_2 \wedge dx_3 + f_2\, dx_3 \wedge dx_1 + f_3\, dx_1 \wedge dx_2.$$

Then

$$\begin{aligned} d\omega &= df_1 \wedge dx_2 \wedge dx_3 + df_2 \wedge dx_3 \wedge dx_1 + df_3 \wedge dx_1 \wedge dx_2 \\ &= \left(\frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} + \frac{\partial f_3}{\partial x_3} \right) dx_1 \wedge dx_2 \wedge dx_3, \end{aligned}$$

or (div $\mathbf{F}$) $dx_1 \wedge dx_2 \wedge dx_3$. This proof is given in more detail in the last section.

Remark 5.3. Note how the classical operators of vector calculus in 3-space (gradient, curl, and divergence) are really the exterior derivative d operator in vector field form. In a similar vein, the condition $d^2 = 0$ can also be shown as equivalent to the famous formulas curl(grad f) $= 0$ and div(curl $\mathbf{F}$) $= 0$, although we will not prove these statements in this paper.

Figure 5.1 gives a visual representation of the exterior derivative operator d in $\mathbb{R}^3$.

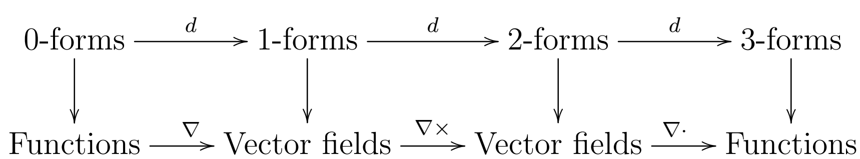

Figure 5.1. Exterior derivative operator d

6. Integration of differential forms

In addition to taking the derivative of differential forms, we can also take the integral of differential forms: in fact, forms were created for integration in that they automatically transform correctly when coordinates change. We now approach integrating differential forms, the last tool we construct before arriving at the ultimate result of the paper, the generalized Stokes' theorem.

6.1. **Integrating differential forms.** Integration in single-variable calculus can be split into three concepts: the indefinite integral $\int f$ (also known as the *anti-derivative*), the unsigned definite integral $\int_{[a,b]} f(x)\,dx$ (used to find area under a curve or mass of a 1D object of varying density), and the signed definite integral $\int_a^b f(x)\,dx$ (which would be used to find work required to move a particle from a to b).

In single-variable calculus, these three integration concepts are rather similar. $F = \int f$ and $\int_a^b f(x)\,dx$ are related by the fundamental theorem of calculus:

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

and signed definite integrals and unsigned definite integrals are related by the simple identity (valid when $a \le b$, flipped when $a \ge b$)

$$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx = \int_{[a,b]} f(x)\,dx.$$

However, when the three integration concepts are moved into multivariable calculus, they begin to diverge significantly from each other: the indefinite integral generalizes to the notion of a solution to a differential equation; the unsigned definite integral generalizes to integration on a measure space; the signed definite integral generalizes to the integration of forms, the focus of this section.

Integrating over differential 1-forms. The first (or among the first) integrals learnt in calculus, the signed definite integral, can be viewed as the integration of a differential 1-form $f(x)\,dx$. The line integral

$$W = \int_C F\,dx + G\,dy + H\,dz$$

is also the integral of a 1-form $(F\,dx + G\,dy + H\,dz)$ except over a curve rather than an interval.
We now generalize to integrating differential k-forms. More details can be found at [Mun91].

Integrating over differential k-forms. First, let us treat the case where the support of ω can be covered by a single coordinate patch.

Definition 6.1. Let M be a compact oriented k-manifold in $\mathbb{R}^n$. Let ω be a k-form defined in an open set of $\mathbb{R}^n$ containing M. Let $C = M \cap (\text{Support}\omega)$; then C is compact. Suppose there is a coordinate patch $\alpha : U \to V$ on M belonging to the orientation of M such that $C \subset V$. By replacing U by a smaller open set if necessary, we can assume that U is bounded. We define the **integral of** ω **over** M by the equation

$$\int_M \omega = \int_{\text{int } U} \alpha^* \omega.$$

Defining Int U quickly: Int $U = U$ if U is open in $\mathbb{R}^k$, and Int $U = U \cap \mathbb{H}^k_+$ if U is open in $\mathbb{H}^k$ but not in $\mathbb{R}^k$. Note that $\mathbb{H}^k$ denotes the upper half-space in $\mathbb{R}^k$, consisting of $x \in \mathbb{R}^k$ for which $x_k \geq 0$; $\mathbb{H}^k_+$ denotes the open upper half-space, consisting of those x for which $x_k > 0$.

To define $\int_M \omega$ in general, we use a partition of unity. Essentially, we split M into small pieces and then integrate over those pieces.

Definition 6.2. Let M be a compact oriented k-manifold in $\mathbb{R}^n$. Let ω be a k-form defined in an open set of $\mathbb{R}^n$ containing M. Cover M by coordinate patches belonging to the orientation of M; then choose a partition of unity $\phi, \ldots, \phi_l$ on M that is dominated by this collection of coordinate patches on M. We define the **integral of** ω **over** M by the equation

$$\int_M \omega = \sum_{i=1}^{l} \left[\int_M \phi_i \omega \right].$$

6.2. **Fubini's Theorem.** Before we give and prove the more general Fubini's theorem, let us review the version that is taught in a typical multivariable calculus class.

Theorem 6.3 (Fubini's Theorem, simpler version)**.** *Let R be the rectangle defined by the inequalities*

$$a \leq x \leq b, \qquad c \leq y \leq d.$$

If $f(x,y)$ is continuous on this rectangle, then

$$\iint\limits_R f(x,y)\, dA = \int_c^d \int_a^b f(x,y)\, dx\, dy = \int_a^b \int_c^d f(x,y)\, dy\, dx.$$

Theorem 6.3 is used to switch the order of integration in double integrals over rectangular regions. We can actually generalize this specific case of Fubini's theorem to other more complex regions.

Theorem 6.4. *Fubini's Theorem, generalized Suppose that a function $f : P \to \mathbb{R}$ is integrable over P. Given a point $x \in P_1$ let us define a function*

$f_x : P_2 \to \mathbb{R}$ by the formula $f_x(y) = f(x,y)$, $y \in P_2$. Then

$$\int_P f\, dV_n = \int_{P_1} \left(\underline{\int}_{P_2} f_x\, dV_{n-k} \right) dV_k = \int_{P_1} \left(\overline{\int}_{P_2} f_x\, dV_{n-k} \right) dV_k.$$

In particular, if the function f_x is integrable for all (or almost all) $x \in P_1$ then one has

$$\int_P f\, dV_n = \int_{P_1} \left(\int_{P_2} f_x\, dV_{n-k} \right) dV_k.$$

Proof. Choose any partition ϕ_1 of P_1 and ϕ_2 of P_2. We will denote elements of the partition ϕ_1 by $P_1^{\boldsymbol{j}}$ and elements of the partition ϕ_2 by $P_2^{\boldsymbol{i}}$. Therefore, products of $P^{\boldsymbol{j},\boldsymbol{i}} = P_1^{\boldsymbol{j}} \times P_2^{\boldsymbol{I}}$ form a partition ϕ of $P = P_1 \times P_2$. Let us denote

$$\bar{I}(x) := \overline{\int}_{P_2} f_x, \quad \underline{I}(x) := \underline{\int}_{P_2} f_x, \quad x \in P_1.$$

Let us now show that

$$L(f,\phi) \le L(\underline{I},\phi_1) \le U(\bar{I},\phi_1) \le U(f,\phi)$$

Note that we have

$$L(f,\phi) = \sum_{\boldsymbol{j}} \sum_{\boldsymbol{I}} m_{\boldsymbol{i},\boldsymbol{j}}(f) \mathrm{Vol}_n P^{\boldsymbol{i},\boldsymbol{j}}. \tag{6.1}$$

The first sum in 6.1 is taken over all multi-indices $\boldsymbol{j}$ of the partition ϕ_1, and the second sum is taken over all multi-indices of $\boldsymbol{i}$ of the partition ϕ_2. On the other hand,

$$L(\underline{I},\phi_1) = \sum_{\boldsymbol{j}} \inf_{x \in P_1^{\boldsymbol{j}}} \left(\underline{\int}_{P_2} f_x\, dV_{n-k} \right) \mathrm{Vol}_k P_1^{\boldsymbol{j}}$$

Note that for every $x \in P_1^{\boldsymbol{j}}$ we have

$$\underline{\int}_{P_2} f_x\, dV_{n-k} \ge L(f_x;\, \phi_2) = \sum_{\boldsymbol{i}} m_{\boldsymbol{i}}(f_x)\, \mathrm{Vol}_{n-k}(P_2^{\boldsymbol{i}}) \ge \sum_{\boldsymbol{i}} m_{\boldsymbol{i},\boldsymbol{j}}(f)\, \mathrm{Vol}_{n-k}(P_2^{\boldsymbol{i}}),$$

and hence

$$\inf_{x \in P_1^{\boldsymbol{j}}} \underline{\int}_{P_2} f_x\, dV_{n-k} \ge \sum_{\boldsymbol{i}} m_{\boldsymbol{i},\boldsymbol{j}}(f)\, \mathrm{Vol}_{n-k}(P_2^{\boldsymbol{i}}).$$

Therefore,

$$L(\underline{I},\phi_1) \ge \sum_{\boldsymbol{j}} \sum_{\boldsymbol{i}} m_{\boldsymbol{i},\boldsymbol{j}}(f) \mathrm{Vol}_{n-k} \left(P_2^{\boldsymbol{i}}\right) \mathrm{Vol}_k \left(P_1^{\boldsymbol{j}}\right) = \sum_{\boldsymbol{j}} \sum_{\boldsymbol{i}} m_{\boldsymbol{i},\boldsymbol{j}}(f) \mathrm{Vol}_n \left(P^{\boldsymbol{i},\boldsymbol{j}}\right) = L(f,\phi).$$

Similarly, one can check that $U(\bar{I},\phi_1) \le U(f,\phi)$. With an obvious inequality $L(\underline{I},\phi_1) \le U(\bar{I},\phi_1)$, we have proved 6.1. Thus, we have

$$\max(U(\bar{I},\phi_1) - L(\bar{I},\phi_1), U(\underline{I},\phi_1) - L(\underline{I},\phi_1)) \le U(\bar{I},\phi_1) - L(\underline{I},\phi_1) \le U(f,\phi) - L(f,\phi)$$

By assumption for appropriate choices of partitions, the right side is $< \epsilon$ *a priori* $\epsilon > 0$. This implies the integrability of the function $\underline{I}(x)$ and $\bar{I}(x)$ over ϕ_1. We can then write

$$\int_{P_1} \underline{I}(x)\, dV_{n-k} = \lim_{\delta(\phi_1)\to 0} L(\underline{I};\phi_1)$$

and

$$\int_{P_1} \bar{I}(x)\, dV_{n-k} = \lim_{\delta(\phi_1)\to 0} U(\bar{I};\phi_1).$$

We also have

$$\lim_{\delta(\phi)\to 0} L(f;\phi) = \lim_{\delta(P)\to 0} U(f;\phi) = \int_P f\, dV_n.$$

Hence, the inequality 6.1 implies that

$$\int_P f\, dV_n = \int_{P_1} \left(\underline{\int}_{P_2} f_x\, dV_{n-k} \right) dV_k = \int_{P_1} \left(\bar{\int}_{P_2} f_x\, dV_{n-k} \right) dV_k.$$

■

Note how a basic idea of "switching" occurs in both versions of Fubini's Theorem.

Remark 6.5. We now progress to the generalized Stokes' theorem.

7. The Generalized Stokes' Theorem

Theorem 7.1. *Let A be any compact-oriented k-dimensional manifold with boundary, so ∂A is a $k-1$ dimensional manifold with the boundary orientation. If ω is any smooth $(k-1)$ form on A, then*

$$\int_{\partial A} \omega = \int_A d\omega$$

Proof. We will prove only the case when A is a manifold with boundary without corners. Feel free to venture into [Mun91] and [GP10] to see other proofs of the theorem for general manifolds.

First, let us cover A by coordinate neighborhoods such that in each neighborhood, A is given either by 3.7 or 3.8. If we can prove the generalized Stokes' theorem holds for each ω_j, then it also holds for ω. Essentially, we are splitting the manifold A into parts and approaching each part (or rather, each type of part) individually as if the parts abide by the generalized Stokes' theorem, so does the whole.

We now need to parameterize ω, as we do for integrating over surfaces in usual calculus classes. Let us assume that ω is supported in one of the coordinate neighborhoods. Consider the corresponding parameterization

$$\phi : G \to U \subset V, \;\; G \subset \mathbb{R}^n$$

with coordinates $u_1, \ldots, u_n$. Note now that $A \cap U = \phi(G \cap L)$, where L is equal to the subspace $\mathbb{R}^k = \{u_{k+1} = \cdots u_n = 0\}$ in the case 3.7 and the upper-half space $\mathbb{R}^k \cap \{u_1 \geq 0\}$. By definition, we then have

$$\int_A d\omega = \int_U d\omega = \int_{G\cap L} \phi^* \, d\omega = \int_{G\cap L} d\phi^* \omega$$

Although the form $\tilde{\omega} = \phi^* \omega\big|_{G\cap L}$ (parameterized ω) is defined only on $G \cap L$, we can extend it to a smooth form on the entirety of L by setting it equal to 0 outside its neighborhood. With this extension, we have

$$\int_{G\cap L} d\tilde{\omega} = \int_L d\tilde{\omega}.$$

The $(k-1)$-form $\tilde{\omega}$ can be written in coordinates $u_1, \ldots, u_k$ as

$$\tilde{\omega} = \sum_1^j f_j(u) \, du_1 \wedge \cdots \overset{j}{\hat{}} \cdots \wedge du_k$$

Taking the integral of the derivative of each side (essentially keeping the equation the "same" by the fundamental theorem of calculus),

$$\int_{G\cap L} d\tilde{\omega} = \int_L \left(\sum_1^k \frac{\partial f_j}{\partial u_j} \right) du_1 \wedge \cdots \wedge du_k.$$

Let us now choose a sufficiently $R > 0$ so that the cube $I = \{|u_i| \leq R, i = 1, \ldots, k\}$ contains the support of ω.

Here, we split the proof to analyze the two cases of submanifolds (pieces of the whole manifold) separately: 3.7 (submanifolds) and 3.8 (submanifolds with boundary).

We begin with case 3.7.

$$\begin{aligned}
\int_{G\cap L} d\tilde{\omega} &= \sum_1^k \int_{\mathbb{R}^k} \frac{\partial f_j}{\partial u_j} \, dV = \sum_1^k \int_{-R}^R \cdots \int_{-R}^R \frac{\partial f_j}{\partial u_j} \, du_1 \cdots du_n \\
&= \sum_k^1 \int_{-R}^R \cdots \left(\int_{-R}^R \frac{\partial f_j}{\partial u_j} \, du_j \right) du_1 \cdots du_{j-1} \, du_{j+1} \cdots du_n \\
&= 0 \qquad (7.1)
\end{aligned}$$

because

$$\int_{-R}^{R} \frac{\partial f_j}{\partial u_j} = f_j(u_1, \ldots, u_{i-1}, R, u_1, \ldots, u_n) - f_j(u_1, \ldots, u_{i-1}, -R, u_1, \ldots, u_n) = 0.$$

Note that in this case, the support of ω does not intersect the boundary of A, and thus $\int_{\partial A} \omega = 0$. Therefore, we have shown that the generalized Stokes' theorem holds in this case.

Moving onto the second case (3.8),

$$\begin{aligned}
\int_{G \cap L} d\tilde{\omega} &= \sum_{1}^{k} \int_{\{u_1 \geq 0\}} \frac{\partial f_j}{\partial u_j} \, dV \\
&= \sum_{1}^{k} \int_{0}^{R} \left(\int_{-R}^{R} \cdots \int_{-R}^{R} \frac{\partial f_j}{\partial u_j} \, du_n \cdots du_2 \right) du_1 \\
&= \int_{-R}^{R} \left(\int_{-R}^{R} \cdots \int_{0}^{R} \frac{\partial f_1}{\partial u_1} \, du_1 \cdots du_{n-1} \right) du_n \\
&= - \int_{-R}^{R} \cdots \int_{-R}^{R} f_1(0, u_2, \ldots, u_n) \, du_2 \cdots du_n. \qquad (7.2)
\end{aligned}$$

Note that all terms in the sum with $j > 1$ are equal to 0 by the same argument as in 7.1.

We now move onto the other side of the generalized Stokes' theorem for this case.

$$\begin{aligned}
\int_{\partial A} \omega = \int_{\{u_1 = 0\}} \phi^* \omega &= \int \int_{\{u_1 = 0\}} f_1(0, u_2, \ldots, u_n) \, du_2 \wedge \cdots \wedge du_n \\
&= - \int_{-R}^{R} \cdots \int_{-R}^{R} f_1(0, u_2, \ldots, u_n) \, du_2 \cdots du_n. \qquad (7.3)
\end{aligned}$$

Remark 7.2. The minus sign is due to the induced orientation on the space $\{u_1 = 0\}$ as the boundary of the upper-half space $\{u_1 \geq 0\}$ is opposite to the orientation defined by the volume form $du_2 \wedge \cdots \wedge du_n$.

Comparing 7.2 and 7.3, we conclude that $\int_A d\omega = \int_{\partial A} \omega$, thus proving the generalized Stokes' theorem.

■

8. Connecting the Generalized Stokes' Theorem with other theorems

We now attempt to end this paper where it started, backtracking from the generalized Stokes' theorem to the more specific cases that are classical theorems in vector calculus after going through the means of generalizing them.

We begin with the $\mathbb{R}^2$ case where ω is a differential 1-form. Let $\omega = P\,dx + Q\,dy$ and D be a region where $C = \partial D$ is the boundary curve.

Calculating $d\omega$:

$$\begin{aligned} d\omega &= d(P\,dx + Q\,dy) = d(P\,dx) + d(Q\,dy) \\ &= (dP) \wedge dx + P \cdot d(dx) + (dQ) \wedge dy + Q \cdot d(dy) \\ &= (P_x\,dx + P_y\,dy) \wedge dx + (Q_x\,dx + Q_y\,dy) \wedge dy \\ &= P_y\,dy \wedge dx + Q_x\,dx \wedge dy \\ &= (Q_x - P_y)\,dx \wedge dy \end{aligned}$$

Restating the generalized Stokes' theorem which we proved earlier:

$$\int_{\partial A} \omega = \int_A d\omega.$$

Plugging the values into the generalized Stokes' theorem,

$$\int_C P\,dx + Q\,dy = \int_D (Q_x - P_y)\,dx \wedge dy,$$

also known as the **Green's Theorem**.

Remark 8.1. Green's Theorem is actually a specific case of the original Stokes' theorem.

Altering the problem slightly, we begin with a differential 2-form ω in $\mathbb{R}^3$. Let $\omega = P\,dy \wedge dz + Q\,dz \wedge dx + H\,dx \wedge dy$ and $G \subset \mathbb{R}^3$ be a domain bounded by a smooth surface S where $S = \partial G$ (G is the *closed solid* enclosed by S).

Again, we calculate $d\omega$:

$$\begin{aligned}
d\omega &= d(P\,dy \wedge dz + Q\,dz \wedge dx + H\,dx \wedge dy) \\
&= (dP)\,dy \wedge dz + (dQ)\,dz \wedge dx + (dH)\,dx \wedge dy \\
&= (P_x\,dx + P_y\,dy + P_z\,dz)\,dy \wedge dz + (Q_x\,dx + Q_y\,dy + Q_z\,dz)\,dz \wedge dx \\
&\quad + (H_x\,dx + H_y\,dy + H_z\,dz)\,dx \wedge dy \\
&= P_x\,dx \wedge dy \wedge dz + Q_y\,dy \wedge dz \wedge dx + H_z\,dz \wedge dx \wedge dy \\
&= (P_x + Q_y + H_z)\,dx \wedge dy \wedge dz \\
&= \text{div } \mathbf{F}
\end{aligned}$$

where $\mathbf{F} = \langle P, Q, H \rangle$.

Plugging values in,

$$\int_S P\,dy \wedge dz + Q\,dz \wedge dx + H\,dx \wedge dy = \int_G \text{div } \mathbf{F},$$

also known as the **Divergence Theorem**.

Altering the problem again, assume ω to be a differential 1-form in $\mathbb{R}^3$. Let $\omega = P\,dx + Q\,dy + H\,dz$ and $S \subset \mathbb{R}^3$ be a surface with boundary curve C ($C = \partial S$).

We will state directly $d\omega$ without proof due to already going through the process in 4.5:

$$d\omega = \text{curl } \mathbf{F}$$

where $\mathbf{F} = \langle P, Q, H \rangle$.

We can now plug values in:

$$\int_C P\,dx + Q\,dy + H\,dz = \int_S \text{curl } \mathbf{F},$$

commonly called the (original) **Stokes' theorem**.

Lastly, we use the generalized Stokes' theorem to derive a result that is *fundamental* to calculus.

Consider the case where ω is a 0-form in $\mathbb{R}^1$. Under these conditions, let ω be some function f and I be the interval $[a, b]$ whilst ∂I consists of the two endpoints, $\{a, b\}$. Assume the orientation of a to be negative and b to be positive.

Applying the generalized Stokes' theorem,

$$\int_{\{-a,b\}} f = \int_{[a,b]} df.$$

Expanding, we get:

$$f(b) - f(a) = \int_a^b f'(x)\,dx,$$

or the **fundamental theorem of calculus** (also called the Newton-Leibniz formula).

9. Conclusion

This paper focused on building the foundational tools up to proving the generalized Stokes' theorem. There are several additional topics regarding the generalized Stokes' theorem and its applications which should be interesting to explore for the reader, such using the theorem to relate integration and mappings. The generalized Stokes' theorem also has many applications in physics.

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