Introduction to Sandpiles

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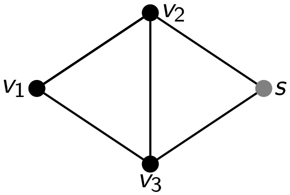
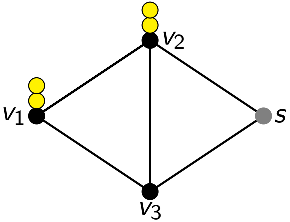
Introduction

Sandpiles were first discussed by Bak, Tang, and Wiesenfeld in 1987 as an example of self-organized criticality. Self-organized criticality is a property of systems that creates complexity through very simple interactions. Since being first discovered, sandpiles have become a big area of mathematical research due to the fascinating properties they have.

We begin by taking a look at this empty graph:



Adding pieces of sand different vertices will give us a *sandpile*:



Sandpile Definition (Continued)

Definition 1.1

A *sandpile* is a graph where each vertex is given a non-negative value that refers to the amount of *sand* or *chips* at that vertex.

Definition 1.2

A vertex is considered *unstable* if the amount of sand at the vertex is greater or equal to the vertex's *degree*.

Definition 1.3

When a vertex is unstable, we *fire* or *topple* sand to its neighbors. What this means is that the unstable vertex will lose its degree amount of sand, each line out of the vertex with distribute one piece of sand to the vertex it is connected to.

Sandpile Definition (Continued)

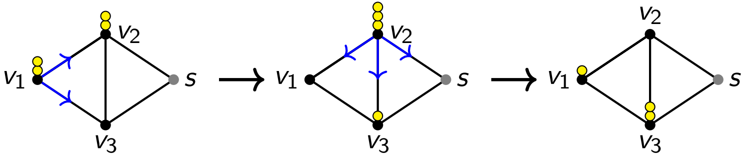
Definition 1.4

The *sink* vertex s is a globally accessible vertex that takes sand and removes it from the graph.

Definition 1.5

We say that a firing of vertex v is *legal* if vertex v is unstable. We define a *firing sequence* to be a sequence of legal firings on numerous vertices.

Usually, our goal is to *stabilize* a sandpile by doing enough vertex firings that we reach a sandpile that has only stable vertices.



Sandpile Properties

The idea of vertex firing and stabilizing automatically brings up some questions. While we won't prove all of them, vertex firings have a lot of very important properties.

1. Vertex firings are commutative.
2. Stabilizing sequences are all rearrangements of each other.
3. All sandpiles with a globally accessible sink have a stabilization.
4. If two firing sequences result in the same sandpile, then the two firing sequences are rearrangements of each other.

A combination of all of these factors is why people call sandpiles the *abelian sandpile model* because it has abelian (commutative) properties.

Sandpile Addition

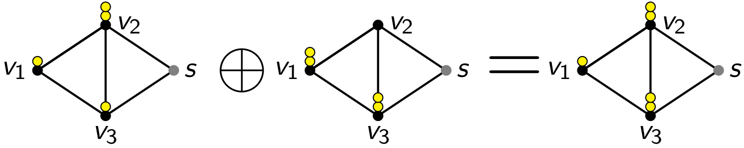
Definition 1.6

The *stable addition* of two sandpiles a and b on graph G, written as $a \oplus b$ denotes the stabilization of $a + b$:

$$a \oplus b = (a + b)^{\circ},$$

where $(a + b)^{\circ}$ represents the stabilization of $a + b$.

Sandpile Addition (Continued)



Sandpile Addition is also associative, which is a product of the uniqueness of stabilization property of sandpiles. If you have three sandpiles a, b, c, $(a + b + c)^\circ = (a \oplus b) \oplus c = a \oplus (b \oplus c)$. because all stabilizations are the same.

Recurrent Sandpiles

Definition 1.7

We define a *recurrent* sandpile c on graph G where for any sandpile a, there is another sandpile b such that $c = (a + b)^{\circ}$.

There is a sandpile that is recurrent for any graph G called the *maximal stable configuration*.

Definition 1.8

The maximal stable configuration $c_{\max}$ is defined as the sandpile where each vertex v has $\deg(v) - 1$ amount of sand on it, thus it is the stable sandpile with the most amount of sand.

Recurrent Sandpiles (Continued)

Theorem 1.9

The maximal stable configuration $c_{\max}$ for a graph G must be recurrent.

Proof.

Let's first look at *stable* sandpile a. By the definition of the maximal stable configuration, we have that $a \leq c_{\max}$ (For a vertex v, the amount of sand at v for c_{max} is greater than or equal to the amount at v for a), thus we can define a sandpile b, where $b(v) = c_{max}(v) - a(v)$. By this definition, we have that $a + b = c_{\max}$. ■

Recurrent Sandpiles (Continued)

Theorem 1.10

A sandpile c is recurrent if and only if there exists a sandpile b such that $c = (c_{\max} + b)^\circ$

Proof.

The forward direction of the theorem is just the definition of recurrent sandpiles, and we already know that $c_{\max}$ is recurrent. The reverse direction takes a little bit more work to show. We know from our initial condition that $c = (c_{\max} + b)^\circ$. Let's take an arbitrary sandpile a. Because $c_{\max}$ is recurrent, we know that there exists sandpile d such that $c_{\max} = (a + d)^\circ$. We will now stable add b to both side and use the uniqueness of a stabilization to finish the proof: $c_{\max} \oplus b = (a + d)^\circ \oplus b \implies c = (a + b + d)^\circ$. Now we know that for any sandpile a, we can find another sandpile such that a plus that sandpile stabilizes to c. Thus, c is recurrent. ■

Recurrent Sandpiles (Continued)

Theorem 1.11

The set of recurrents with the operation being sandpile addition forms a group.

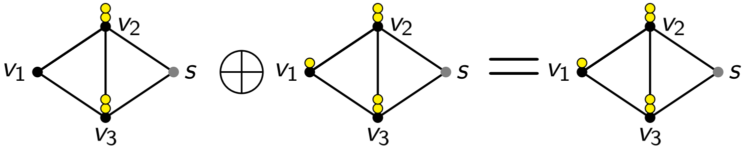
A group is roughly defined as a set with an associative operation $*$ where there is an identity element e where for all elements g we have $e * g = g = g * e$. Also, every element must have an inverse g^{-1} where $g * g^{-1} = e$.

The Identity Sandpile

Definition 1.12

The *identity sandpile* for the set of recurrents is:

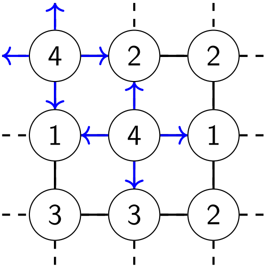
$$(2c_{\max} - (2c_{\max})^{\circ})^{\circ}.$$



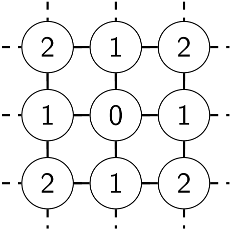
Grid Sandpiles

Definition 1.13

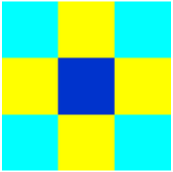
A grid sandpile is a grid of size $m \times n$ where the sink is connected once to all of the vertices on the edge on the boundary other than the corners and twice to the corners.



Grid Sandpiles (Continued)



Above is the identity sandpile for the 3×3 grid. We can assign colors to each amounts of sand to create an image of that sandpile.

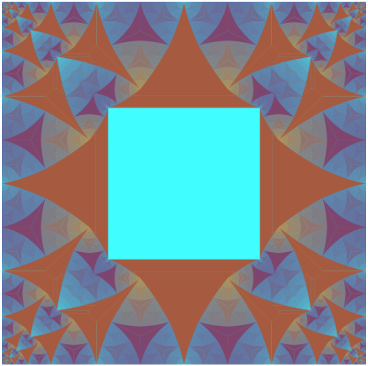


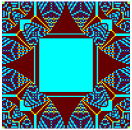
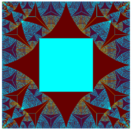
$0 =$ ■, $1 =$ ■, $2 =$ ■, $3 =$ ■

Sandpile Images

This leads us to the final part of this presentation where we will look at identity sandpiles for very large grids and see the amazing fractal patterns. This relates to the *self-organized criticality* discussed earlier where simple systems can create complexity.

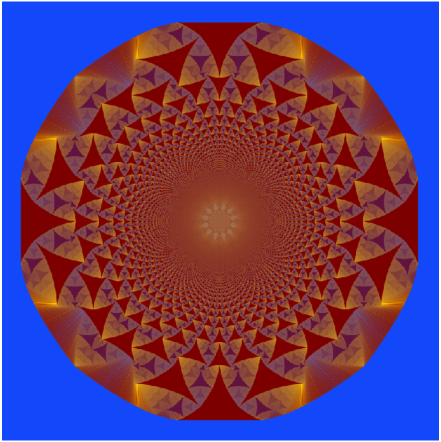
Sandpile Images (Continued)



Identity element for grids of sizes 4000×4000, 100×100, 500×500

Sandpile Images (Continued)



2^{30} pieces of sand dropped at the center of an infinite grid (Wesley Pegden)

Thank You

Thank you all for listening to my talk! If there are any questions, message me on the discord.