# Rogers-Ramanujan Identities

Jinfei Huang
jhuang929@student.fuhsd.org

Jinfei Circle

July 11, 2022

# Integer partitions

**Definition**

A *partition* $\lambda$ is a tuple of positive integers $(\lambda_1, \lambda_2, \ldots, \lambda_k)$ in nonincreasing order, i.e. $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k > 0$. If $\lambda_1 + \cdots + \lambda_k = n$, we say $\lambda$ is a *partition of n*, and write $|\lambda| = n$. We call the $\lambda_i$'s the *parts* of $\lambda$. Let $\ell(\lambda)$ be the number of parts, in this case $k$.

# Integer partitions

**Definition**

A *partition* $\lambda$ is a tuple of positive integers $(\lambda_1, \lambda_2, \ldots, \lambda_k)$ in nonincreasing order, i.e. $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k > 0$. If $\lambda_1 + \cdots + \lambda_k = n$, we say $\lambda$ is a *partition of* $n$, and write $|\lambda| = n$. We call the $\lambda_i$'s the *parts* of $\lambda$. Let $\ell(\lambda)$ be the number of parts, in this case $k$.

**Example**

$\lambda = (8, 5, 4, 3, 2, 2, 1)$ is a partition of 25. We write $8 + 5 + 4 + 3 + 2 + 2 + 1$ for $(8, 5, 4, 3, 2, 2, 1)$.

# Sets of partitions

**Definition**

Let $\mathcal{P}(n)$ denote the set of all partitions of $n$.

# Sets of partitions

## Definition

Let $\mathcal{P}(n)$ denote the set of all partitions of $n$.

## Example

$$\begin{aligned}5 &= 5\\ &= 4+1\\ &= 3+2\\ &= 3+1+1\\ &= 2+2+1\\ &= 2+1+1+1\\ &= 1+1+1+1+1\end{aligned}$$

Thus, $|\mathcal{P}(5)| = 7$.

# Young diagrams

Young diagrams are one of the main ways to represent partitions graphically.

The conjugate of a partition $\lambda$ is the partition given by reflecting its young diagram $[\lambda]$ across the line $y = -x$.

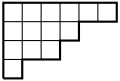


Figure: $\lambda = (6, 4, 3, 1)$

# Young diagrams

Young diagrams are one of the main ways to represent partitions graphically.
The conjugate of a partition $\lambda$ is the partition given by reflecting its young diagram $[\lambda]$ across the line $y = -x$.

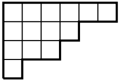


Figure: $\lambda = (6, 4, 3, 1)$

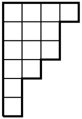


Figure: $\lambda' = (4, 3, 3, 2, 1, 1)$

We see that the conjugate of a partition is also a well-defined partition.
Also, conjugation is an involution on the set of partitions.

# Classes of partitions

**Definition**

Let $\mathcal{D}(n)$ denote the set of partitions of $n$ having distinct parts, and $\mathcal{D} = \bigcup_n \mathcal{D}(n)$.

# Classes of partitions

**Definition**

Let $\mathcal{D}(n)$ denote the set of partitions of $n$ having distinct parts, and $\mathcal{D} = \bigcup_n \mathcal{D}(n)$.

**Definition**

Let $\mathcal{B}(n)$ be the set of partitions of $n$ with parts that differ by at least 2, let $\mathcal{B}'(n)$ be the set of partitions of $n$ with parts differing by at least 2 and smallest part at least 2. Define $\mathcal{B} = \bigcup_n \mathcal{B}(n)$ and $\mathcal{B}' = \bigcup_n \mathcal{B}'(n)$.

# Classes of partitions

### Definition

Let $\mathcal{D}(n)$ denote the set of partitions of $n$ having distinct parts, and $\mathcal{D} = \bigcup_n \mathcal{D}(n)$.

### Definition

Let $\mathcal{B}(n)$ be the set of partitions of $n$ with parts that differ by at least 2, let $\mathcal{B}'(n)$ be the set of partitions of $n$ with parts differing by at least 2 and smallest part at least 2. Define $\mathcal{B} = \bigcup_n \mathcal{B}(n)$ and $\mathcal{B}' = \bigcup_n \mathcal{B}'(n)$.

### Definition

Define $\mathcal{P}_{1,4}(n) = \{\lambda \in \mathcal{P}(n) : \lambda_i \equiv 1, 4 \pmod 5\}$ and $\mathcal{P}_{2,3}(n) = \{\lambda \in \mathcal{P}(n) : \lambda_i \equiv 2, 3 \pmod 5\}$.

# Rogers-Ramanujan identities

**Theorem**

*For all* $n \geq 1$, $|\mathcal{B}(n)| = |\mathcal{P}_{1,4}(n)|$.

**Theorem**

*For all* $n \geq 1$, $|\mathcal{B}'(n)| = |\mathcal{P}_{2,3}(n)|$.

We will only focus on the first identity.

# Example

$$\begin{aligned} 11 &= 11 \\ &= 10 + 1 \\ &= 9 + 2 \\ &= 8 + 3 \\ &= 7 + 4 \\ &= 7 + 3 + 1 \\ &= 6 + 4 + 1 \end{aligned}$$

$$\begin{aligned} 11 &= 11 \\ &= 9 + 1 + 1 \\ &= 6 + 4 + 1 \\ &= 6 + 1 + 1 + 1 + 1 + 1 \\ &= 4 + 4 + 1 + 1 + 1 \\ &= 4 + 1 + 1 + 1 + 1 + 1 + 1 + 1 \\ &= 1 + 1 + \cdots + 1 \end{aligned}$$

$|\mathcal{B}(11)| = 7.$

$|\mathcal{P}_{1,4}(11)| = 7$

# History

- Rogers: proved in 1894, obscure paper

# History

- Rogers: proved in 1894, obscure paper
- Ramanujan: rediscovered circa 1913, sent to Hardy

# History

- Rogers: proved in 1894, obscure paper
- Ramanujan: rediscovered circa 1913, sent to Hardy
- Rogers & Ramanujan (1917): joint proof

# History

- Rogers: proved in 1894, obscure paper
- Ramanujan: rediscovered circa 1913, sent to Hardy
- Rogers & Ramanujan (1917): joint proof
- Schur: rediscovered and proved in 1917

# History

- Rogers: proved in 1894, obscure paper
- Ramanujan: rediscovered circa 1913, sent to Hardy
- Rogers & Ramanujan (1917): joint proof
- Schur: rediscovered and proved in 1917
- Slater (1952): list of 130 Rogers-Ramanujan type identities

# History

- Rogers: proved in 1894, obscure paper
- Ramanujan: rediscovered circa 1913, sent to Hardy
- Rogers & Ramanujan (1917): joint proof
- Schur: rediscovered and proved in 1917
- Slater (1952): list of 130 Rogers-Ramanujan type identities
- Garsia and Milne (1981): first bijective proof with involution principle

# Analytic form

We can translate the main result into a statement about the relevant generating functions.

$$1+\sum_{k=1}^{\infty}\frac{q^{k^2}}{(1-q)(1-q^2)\cdots(1-q^k)}=\prod_{n=0}^{\infty}\frac{1}{(1-q^{5n+1})(1-q^{5n+4})}.$$

$$1+\sum_{k=1}^{\infty}\frac{q^{k^2+k}}{(1-q)(1-q^2)\cdots(1-q^k)}=\prod_{n=0}^{\infty}\frac{1}{(1-q^{5n+2})(1-q^{5n+3})}.$$

# Jacobi triple product

We can use Jacobi's classic triple product identity to rewrite the product side of the first identity.

Theorem

$$\sum_{m=-\infty}^{\infty} z^m q^{m^2} = \prod_{n=0}^{\infty}(1 + zq^{2n+1})(1 + z^{-1}q^{2n+1})(1 - q^{2n+2}).$$

# Schur's identity

By substituting $(q, z) \mapsto (t^{5/2}, -t^{1/2})$, we have

$$\sum_{m=-\infty}^{\infty} (-1)^m t^{\frac{m(5m+1)}{2}} = \prod_{n=0}^{\infty} (1 - t^{5n+2})(1 - t^{5n+3})(1 - t^{5n+5}),$$

$$\prod_{j=1}^{\infty} \frac{1}{1 - t^j} \sum_{m=-\infty}^{\infty} (-1)^m t^{\frac{m(5m+1)}{2}} = \prod_{n=0}^{\infty} \frac{1}{(1 - t^{5n+1})(1 - t^{5n+4})}.$$

By the Jacobi triple product, the first Rogers-Ramanujan identity is equivalent to the following:

$$\prod_{j=1}^{\infty} (1 - t^j) \left( 1 + \sum_{k=1}^{\infty} \frac{t^{k^2}}{(1 - t)(1 - t^2) \cdots (1 - t^k)} \right) = \sum_{m=-\infty}^{\infty} (-1)^m t^{\frac{m(5m+1)}{2}}.$$

# Combinatorial interpretation

Equivalently,

$$\sum_{\lambda \in \mathcal{D}} (-1)^{\ell(\lambda)} t^{|\lambda|} \sum_{\mu \in \mathcal{B}} t^{|\mu|} = \sum_{m=-\infty}^{\infty} (-1)^m t^{\frac{m(5m+1)}{2}},$$

$$\sum_{(\lambda,\mu) \in \mathcal{D} \times \mathcal{B}} (-1)^{\ell(\lambda)} t^{|\lambda|+|\mu|} = \sum_{m=-\infty}^{\infty} (-1)^m t^{\frac{m(5m+1)}{2}}.$$

Set $\mathcal{R} = \mathcal{D} \times \mathcal{B}$, and $\mathcal{R}_n = \{(\lambda, \mu) \in \mathcal{R} : |\lambda| + |\mu| = n\}$. The *sign* of a pair $(\lambda, \mu)$ is the parity of $\ell(\lambda)$. We want to define an involution $\alpha$ on $R_n$ that is sign-reversing for everything except the fixed points of the involution.

# $g(\lambda)$ and $u(\mu)$

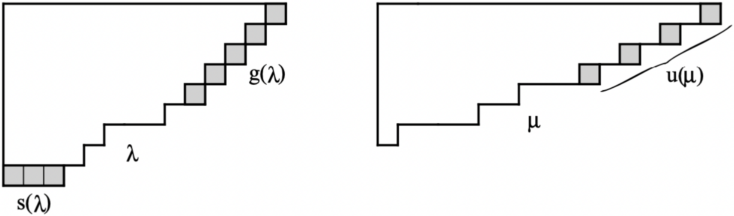
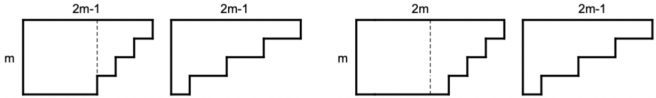



FIGURE 30. For a pair of partitions $(\lambda, \mu) \in \mathcal{R}$ as above, we have $s(\lambda) = 3$, $g(\lambda) = 5$, $u(\mu) = 4$.

We abbreviate $s = s(\lambda)$, $g = g(\lambda)$, $u = u(\mu)$.

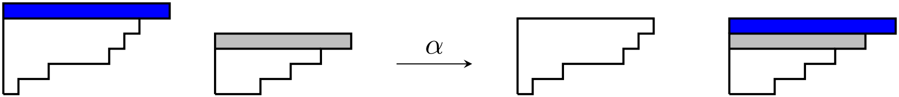
# Schur's involution

Let $\mathcal{F}$ be the set of fixed points of $\alpha$, defined in the following diagram.




Start with $(\lambda, \mu) \in \mathcal{R}_n$, suppose this is not a fixed point.
First, compare $\lambda_1$ and $\mu_1$. If $\lambda_1 \geq \mu_1 + 2$, move part $\lambda_1$ to $\mu$. If $\lambda_1 < \mu_1$, move part $\mu_1$ to $\lambda$.

# Example 1



# Schur's involution cont.

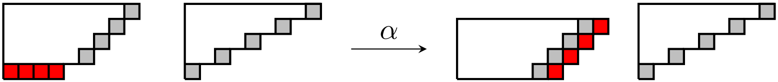
There remain the cases $\lambda_1 = \mu_1$ and $\lambda_1 = \mu_1 + 1$. Denote these cases by $\mathcal{R}_n^1$ and $\mathcal{R}_n^2$, respectively.

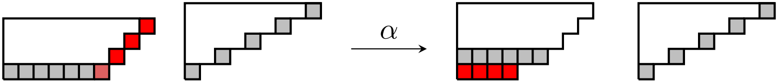
If $(\lambda, \mu) \in \mathcal{R}_n^1$ and $s \leq g, u$, move $s$ to $g$. Conversely, if $(\lambda, \mu) \in \mathcal{R}_n^2$, $g < s, g \leq u$, move $g$ to $s$.

If $(\lambda, \mu) \in \mathcal{R}_n^1$ and $g < s, u$, move $g$ to $u$ and attach $\mu_1$ to $\lambda$. Conversely, if $(\lambda, \mu) \in \mathcal{R}_n^2$, and $u < s, g$, move $u$ to $g$ and attach $\lambda_1$ to $\mu$.

If $(\lambda, \mu) \in \mathcal{R}_n^1$ and $g < s, u$, move $g$ to $u$ and attach $\mu_1$ to $\lambda$. Conversely, if $(\lambda, \mu) \in \mathcal{R}_n^2$, and $u < s, g$, move $u$ to $g$ and attach $\lambda_1$ to $\mu$.

# Example 2



# Proving Schur's identity

**Claim**

*The map $\alpha : \mathcal{R}_n \to \mathcal{R}_n$ is an involution which is sign-reversing except for the fixed points.*

# Proving Schur's identity

**Claim**

*The map $\alpha : \mathcal{R}_n \to \mathcal{R}_n$ is an involution which is sign-reversing except for the fixed points.*

**Proof.**

(Sketch:) Start with $(\lambda, \mu)$, and let $(\hat{\lambda}, \hat{\mu})$ be its image under $\alpha$. If $(\lambda, \mu) \in \mathcal{R}_n$, i.e. $|\lambda| + |\mu| = n$, then $|\hat{\lambda}| + |\hat{\mu}| = n$. One can verify that $\alpha = \alpha^{-1}$ by casework. Moreover, $\alpha$ always changes $\ell(\lambda)$ by 1, so $\alpha$ is indeed sign-reversing.

# Proving Schur's identity

**Claim**

*The map $\alpha : \mathcal{R}_n \to \mathcal{R}_n$ is an involution which is sign-reversing except for the fixed points.*

**Proof.**

(Sketch:) Start with $(\lambda, \mu)$, and let $(\hat{\lambda}, \hat{\mu})$ be its image under $\alpha$. If $(\lambda, \mu) \in \mathcal{R}_n$, i.e. $|\lambda| + |\mu| = n$, then $|\hat{\lambda}| + |\hat{\mu}| = n$. One can verify that $\alpha = \alpha^{-1}$ by casework. Moreover, $\alpha$ always changes $\ell(\lambda)$ by 1, so $\alpha$ is indeed sign-reversing.

For every $(\lambda, \mu) \in \mathcal{R}_n \setminus \mathcal{F}$, we pair it up with $(\hat{\lambda}, \hat{\mu})$ (because $\alpha$ is an involution). The two terms cancel out in the sum:

$$(-1)^{\ell(\lambda)} t^{|\lambda|+|\mu|} + (-1)^{\ell(\hat{\lambda})} t^{|\hat{\lambda}|+|\hat{\mu}|} = (-1)^{\ell(\lambda)} t^{|\lambda|+|\mu|} - (-1)^{\ell(\lambda)} t^{|\lambda|+|\mu|} = 0.$$

# Proving Schur's identity, cont.

Therefore, the LHS of Schur's identity becomes

$$\begin{aligned}\sum_{(\lambda,\mu)\in\mathcal{R}} (-1)^{\ell(\lambda)} t^{|\lambda|+|\mu|} &= \sum_{(\lambda,\mu)\in\mathcal{F}} (-1)^{\ell(\lambda)} t^{|\lambda|+|\mu|} \\ &= 1 + \sum_{m=1}^{\infty} (-1)^m t^{m(5m-1)/2} + \sum_{m=1}^{\infty} (-1)^m t^{m(5m+1)/2} \\ &= \sum_{m=-\infty}^{\infty} (-1)^m t^{m(5m+1)/2}.\end{aligned}$$

which finishes the proof.

# Thank you