THE ROGERS-RAMANUJAN IDENTITIES

JINFEI HUANG

ABSTRACT. We present an elementary combinatorial proof of the first Rogers-Ramanujan identity with the help of a generalized Dyson's rank. We then prove the second identity using a variant of Schur's involution. A connection to Ramanujan's continued fraction is demonstrated.

CONTENTS

1. INTRODUCTION

The Rogers-Ramanujan identities have been regarded as the most beautiful pair of identities in mathematics. We wish to present a case for the veracity of this evaluation.

There are two main forms of the Rogers-Ramanujan identity. On the one hand, they can be expressed as the following product-sum identities:

$$\sum_{k=1}^{\infty} \frac{q^{k^2}}{(1-q)(1-q^2)\cdots(1-q^k)} = \prod_{n=0}^{\infty} \frac{1}{(1-q^{5n+1})(1-q^{5n+4})}, \tag{1.1}$$

$$\sum_{k=1}^{\infty} \frac{q^{k^2+k}}{(1-q)(1-q^2)\cdots(1-q^k)} = \prod_{n=0}^{\infty} \frac{1}{(1-q^{5n+2})(1-q^{5n+3})}. \tag{1.2}$$

where we set $(1-q)(1-q^2)\cdots(1-q^k) = 1$ when $k = 0$ for convenience. Equivalently, we can formulate them in words: the first identity (1.1) states that the number of partitions of an integer $n \geq 0$ into distinct nonconsecutive parts by at least 2 is the same as the number of partitions of n into parts congruent to 1 or 4 (mod 5). The second identity (1.2) is similar, stating that the number of partitions of n into distinct nonconsecutive parts each strictly greater than 1 is the same as the number of partitions of n with parts congruent to 2 or 3 (mod 5). For example, we

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$$\begin{aligned}11 &= 11 \\ &= 10+1 \\ &= 9+2 \\ &= 8+3 \\ &= 7+4 \\ &= 7+3+1 \\ &= 6+4+1\end{aligned} \qquad \begin{aligned}11 &= 11 \\ &= 9+1+1 \\ &= 6+4+1 \\ &= 6+1+1+1+1+1 \\ &= 4+4+1+1+1 \\ &= 4+1+1+1+1+1+1+1 \\ &= 1+1+\cdots+1\end{aligned}$$

Figure 1. Partitions of 11 with parts differing by at least 2 on the left, partitions of 11 with parts congruent to 1 or 4 (mod 5) on the right. There are 7 of each. The last partition on the right consists of 11 parts of 1.

can empirically verify the first Rogers-Ramanujan identity for $n = 11$ (see Figure 1). A formal treatment of partition terminology we will put off until Section 2.

L. J. Rogers was the first to discover (1.1) and (1.2); however, Rogers lived in obscurity and his 1894 paper was largely ignored. The identities resurfaced with Ramanujan's rediscovery some time before 1913. He sent them to Hardy, and in typical Ramanujan style, he did not provide a proof with his results. It went on to be published. Then in 1917, Ramanujan stumbled upon Rogers's 1894 paper. Curiously, this was not his first time reading the paper, as he had previously seen the identities but they "had entirely slipped from [his] memory."

Rogers and Ramanujan coauthored a joint proof of the two identities. Meanwhile, I. Schur independently rediscovered and proved them in 1917. It is generally agreed upon that Schur's combinatorial proof is fundamentally different from the aforementioned collaboration [And89]. One might then be inclined to wonder why (1.1) and (1.2) are not attributed to Schur. Indeed, neither Rogers nor Ramanujan considered their combinatorial interpretations. The now-standard name "Rogers-Ramanujan identities" is due to Hardy, who has been criticized for his "tabloid sensationalism" and the "historical injustice" of the name [**?**].

Historically, analytic proofs of identities in partition theory typically preceded combinatorial proofs, which were more difficult to find. In fact, a direct bijective proof of the Rogers-Ramanujan identities has yet to be discovered, where a direct bijection is a one that can be constructed without any intermediate steps [Pak06]. The lack of a direct bijective proof is not unique to the Rogers-Ramanujan identities, although one may argue that such a proof ought to exist for "important" partition identities.

In the present paper, we emphasize combinatorial methods and make heavy use of known bijections for a few reasons. Analytic proofs of the Rogers-Ramanujan identities, such as the motivated proof given by Andrews and Baxter [AB89], are generally more accessible to a wider audience. Moreover, some of the combinatorial ideas naturally give rise to definitions and examples using diagrams, the explanatory power of which is obvious.

Acknowledgements

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2. Preliminaries

2.1. **Notation.** We define a *partition* λ to be a k-tuple of positive integers $(\lambda_1, \lambda_2, \ldots, \lambda_k)$ in nonincreasing order, i.e. $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k > 0$. We say λ is a partition of n, or $|\lambda| = n$, if $\lambda_1 + \lambda_2 + \cdots + \lambda_k = n$; call $\lambda_1, \ldots, \lambda_k$ the *parts* of λ. Let $\ell(\lambda) = k$ be the number of parts, and $\lambda_j = 0$ for all $j > k$. By a slight abuse of notation, we will sometimes write $\lambda_1 + \lambda_2 + \cdots + \lambda_k$ in place $(\lambda_1, \lambda_2, \ldots, \lambda_k)$, identifying a partition with the sum of its parts.

Superscripts indicate how many times an integer occurs as a part in a partition. For instance, one may write $(3, 2, 2, 2) = (3, 2^3)$ or $(3, 1) = (3^1, 2^0, 1^1)$.

We often represent a partition graphically by its corresponding *Young diagram*, which is a collection of unit squares on a square grid. For any $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$, the Young diagram $[\lambda]$ is constructed as follows: construct a row of λ_i left-aligned unit squares in the i-th column from the top, for each $1 \leq i \leq k$. (Alternatively, replacing the squares with nodes produces a so-called *Ferrers graph* [SF82]). The *Durfee square* of a partition λ is the largest square that can fit in the Young diagram $[\lambda]$. More generally, a *Durfee m-rectangle* is the largest rectangle that can fit in $[\lambda]$ such that the difference between the height and width of the rectangle is m. We allow the width of a Durfee m-rectangle to be zero, but the height must always be positive. The Durfee m-rectangle of λ, if well-defined, is always placed in the top-left corner of $[\lambda]$. The concept is best explained with an example (see Figure **??**).

There are two natural ways to combine partitions to form a new partition. Let $\alpha = (\alpha_1^{a(\alpha_1)}, \ldots, 2^{a(2)}, 1^{a(1)})$ and $\beta = (\beta_1^{b(\beta_1)}, \ldots, 2^{b(2)}, 1^{b(1)})$ be partitions, where $a(i)$ (resp. $b(i)$) is the number of occurences of i as an entry in α (resp. β). The union $\alpha \cup \beta$ is the partition

$$\alpha \cup \beta = (M^{a(M)+b(M)}, \ldots, 2^{a(2)+b(2)}, 1^{a(1)+b(1)})$$

where $M = \max\{\alpha_1, \beta_1\}$. Informally, $\alpha \cup \beta$ is simply the combination of all the parts of α and β with some rearrangement if necessary. This definition differs from the union of sets in that repeated parts do not become a single part. Observe that if $\alpha \cup \lambda = \beta \cup \lambda$, then $\alpha = \beta$; this follows easily by the definition of union. Similarly, define the sum

$$\alpha + \beta = (\alpha_1 + \beta_1, \alpha_2 + \beta_2, \ldots, \alpha_{\max\{\ell(\alpha), \ell(\beta)\}} + \beta_{\max\{\ell(\alpha), \ell(\beta)\}})$$

by the pairwise addition of α_i and β_i.

First defined by Dyson to give combinatorial interpretations of Ramanujan's congruences [Dys44], the *rank* of a partition λ is the largest part minus the number of parts, i.e. $\lambda_1 - \ell(\lambda)$. There are multiple useful generalizations of Dyson's rank, such as Garvan's generalization, successive rank, and $(2, m)$ rank; we will only consider the $(2, m)$ rank, relevant in Section 5.

For the purposes of this paper, we assume that all power series and their variables are formal to avoid issues of convergence. In particular, we interpret the expression $1/(1-q)$ as the geometric series

$$\frac{1}{1-q} = 1 + q + q^2 + q^3 + q^4 + \cdots \tag{2.1}$$

Let $p(n)$ denote the number of partditions of n, $d(n)$ denote the number of partitions of n into distinct parts, and $p_k(n)$ denote the number of partitions of n with at most k parts. We take $p(0) = d(0) = p_k(0) = 1$ for convenience, and adopt this convention for partition counting functions

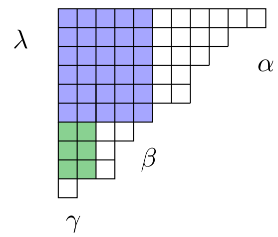


Figure 2. Partition $\lambda = (11, 9, 8, 7, 7, 5, 4, 3, 3, 1)$ and the sum of its parts is $|\lambda| = 58$. The first and second Durfee m-rectangles are shaded in blue and green, respectively, for $m = 1$; we have $s_{2,m}(\lambda) = 6)$ and $t_{2,m}(\lambda) = 3)$. Also, $\alpha = (6, 4, 3, 2, 2)$, $\beta = (2, 1, 1)$, and $\gamma = (1)$.

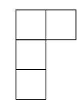


Figure 3. $\lambda = (2, 1, 1)$

in general. Their corresponding generating functions have the well-known formulae:

$$\sum_{n=0}^{\infty} p(n)\, q^n = \prod_{j=1}^{\infty} \frac{1}{1-q^j} \tag{2.2}$$

$$\sum_{n=0}^{\infty} d(n)\, q^n = \prod_{j=1}^{\infty} (1+q^j) \tag{2.3}$$

$$\sum_{n=0}^{\infty} p_k(n)\, q^n = \prod_{j=1}^{k} \frac{1}{1-q^j} \tag{2.4}$$

Let $\mathcal{P}$ be the set of all partitions, $\mathcal{D}$ be the set of all partitions with distinct parts, and $\mathcal{Q}$ be the set of partitions with parts differing by at least 2. We define $C_1(n)$ as the number of partitions of n with no repeated or consecutive parts; similarly, we define $C_2(n)$ as the number of such partitions with parts no less than 2; for instance, $C_1(9) = 5$ and $C_2(9) = 3$.

9	9
$8+1$	$7+2$
$7+2$	$6+3$
$6+3$	
$5+3+1$	

The analytic form of the Rogers-Ramanujan identities can be concisely expressed with the help of some standard shorthand notation:

Definition 2.1. For $n \geq 1$, the *q-Pochhammer symbol* is given by

$$(a;q)_n = (1-a)(1-aq)(1-aq^2)\cdots(1-aq^{n-1})$$

$$(a;q)_\infty = (1-a)(1-aq)(1-aq^2)\cdots,$$

and let $(a;q)_0 = 1$ for convenience. We abbreviate $(a;q)_n$ as $(a)_n$ when context implies the second entry in $(a;q)_n$ is q; similarly, $(a)_\infty := (a;q)_\infty$. Thus, we have

$$\sum_{k=0}^{\infty} \frac{q^{k^2}}{(q;q)_k} = \frac{1}{(q;q^5)_\infty (q^4;q^5)_\infty} \tag{2.5}$$

$$\sum_{k=0}^{\infty} \frac{q^{k^2+k}}{(q;q)_k} = \frac{1}{(q^2;q^5)_\infty (q^3;q^5)_\infty}. \tag{2.6}$$

We employ the well-known Jacobi triple product to rewrite the product sides of (2.5) and (2.6).

Theorem 2.2. (Jacobi triple product)

$$\sum_{m=-\infty}^{\infty} z^m q^{m^2} = (-zq;q^2)_\infty (-z^{-1}q;q^2)_\infty (q^2;q^2)_\infty = \prod_{n=0}^{\infty} (1+zq^{2n+1})(1+z^{-1}q^{2n+1})(1-q^{2n+2}).$$

Proof. Either omitted or sent to an appendix, depending the amount of other content. ∎

We have the following two corollaries of Theorem 2.2:

$$\sum_{m=-\infty}^{\infty} (-1)^m t^{\frac{m(5m+1)}{2}} = \prod_{n=0}^{\infty} (1-t^{5n+2})(1-t^{5n+3})(1-t^{5n+5}) \tag{2.7}$$

$$\sum_{m=-\infty}^{\infty} (-1)^m t^{\frac{m(5m+3)}{2}} = \prod_{n=0}^{\infty} (1-t^{5n+1})(1-t^{5n+4})(1-t^{5n+5}), \tag{2.8}$$

corresponding to the substitutions $(q, z \mapsto t^{5/2}, -t^{1/2})$ and $(q, z \mapsto t^{5/2}, -t^{3/2})$, respectively. Divide both sides of (2.7) by $1/(q;q)_\infty$; the following identity is equivalent to (2.5):

Theorem 2.3.

$$\sum_{k=0}^{\infty} \frac{q^{k^2}}{(q;q)_k} = \frac{1}{(q;q)_\infty} \left(\sum_{m=-\infty}^{\infty} (-1)^m q^{\frac{m(5m+1)}{2}} \right)$$

3. Motivation

The goal of this section is to provide insight into how Ramanujan's celebrated identities may have been naturally discovered and conjectured. Let us introduce some additional notation for this section; define

$$G(q) = \sum_{k=1}^{\infty} \frac{q^{k^2}}{(q;q)_k} \qquad H(q) = \sum_{k=1}^{\infty} \frac{q^{k^2+k}}{(q;q)_k}.$$

to be the sum sides of the first and second Rogers-Ramanujan identities, respectively. We first connect $G(q)$ and $H(q)$ to the q-analog of the simplest possible continued fraction, namely

$$c(q) := 1 + \cfrac{q}{1 + \cfrac{q^2}{1 + \cfrac{q^3}{1 + \cdots}}} \tag{3.1}$$

The *q-analog* of an expression is a generalization of that expression parameterized by a variable q that reduces to the original expression as $q \to 1^-$. There exist q-analogs for numerous mathematical objects, from factorials to nonnegative integers. Similar to how the geometric sum $1+q+\cdots+q^{n-1} =$

$(1-q^n)/(1-q)$ is a q-analog of $1+1+\cdots+1=n\in\mathbb{Z}_{\geq 0}$, $c(q)$ is a q-analog of the continued fraction

$$c(1)=1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cdots}}}$$

We present some notable properties of $c(1)$ before we proceed, as we will analyze its q-analog in an similar manner. Clearly, we have the recurrence $c(1)=1+1/c(1)$; it follows that $c(1)$ is the golden ratio. Furthermore, the truncated fractions of $c(1)$ follow a remarkable pattern.

$$c_1=1=\frac{1}{1}\qquad c_2=1+\frac{1}{1}=\frac{2}{1}\qquad c_3=1+\cfrac{1}{1+\cfrac{1}{1}}=\frac{3}{2}\qquad c_4=1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1}}}=\frac{5}{3}\ \cdots$$

Indeed, the n-th truncated fraction is equal to the ratio of the n-th Fibonacci number and the $(n-1)$-th Fibonacci number. More importantly, note that the denominator of each truncated fraction is the numerator of the previous one (when both are expressed in lowest terms). Does this hold in general for $c(q)$?

Before we answer that question, let us address another concern: is $c(q)$ a sufficiently "nice" generalization of $c(1)$? We would like $c(q)$ to satisfy a simple recurrence like $c(1)=1+1/c(1)$. Wishful thinking may lead us to conjecture $c(q)=1+1/c(q)$, but the powers of q do not match up. We may stubbornly try to force the powers of q to match up anyway, and come up with a new continued fraction:

$$c(z,q)=1+\cfrac{zq}{1+\cfrac{zq^2}{1+\cfrac{zq^3}{1+\cdots}}}$$

Introducing the parameter z allows us to adjust the powers of q by replacing z with, say, zq or zq^{-1}. By definition, $c(z,q)$ satisfies

$$c(z,q)=1+\frac{zq}{c(zq,q)}$$

Now let $c_n(z,q)$ be the n-th truncated fraction of $c(z,q)$, defined similarly to the c_n; let $H_n(z,q)$ be the polynomial numerator of $c_n(z,q)$. Computing $c_n(z,q)$ for $n=1,2,3,4$

$$\begin{aligned}
c_1(z,q)&=1=1;\\
c_2(z,q)&=1+\frac{zq}{1}=1+zq\\
c_3(z,q)&=1+\cfrac{zq}{1+\cfrac{zq^2}{1}}=\frac{1+zq+zq^2}{1+zq^2}\\
c_4(z,q)&=1+\cfrac{zq}{1+\cfrac{zq^2}{1+\cfrac{zq^3}{1}}}=\frac{1+zq+zq^2+zq^3+z^2q^4}{1+zq^2+zq^3}
\end{aligned}$$

suggests that the denominator of $c_n(z,q)$ is $H_{n-1}(zq,q)$ for $n>1$; i.e., the denominator of $c_n(z,q)$ is the numerator of $c_{n-1}(z,q)$ up to a change of variables $z\mapsto zq$. Thus, we substitution $c(z,q)=$

$H(z,q)/H(zq,q)$ for some power series $H(z,q)$ (under the assumption that a power series exists). This gives

$$H(z,q) = H(zq,q) + zqH(zq^2,q) \tag{3.2}$$

We write $H(z,q) = \sum_{n=0}^{\infty} h_n(q) z^n$. After rewriting the sums in (3.2) and comparing coefficients, we have

$$h_n(q) = \frac{q^{2n-1}}{1-q^n}\, h_{n-1}(q) \tag{3.3}$$

for all $n \geq 1$; repeated application of (3.3) yields

$$h_n(q) = \frac{q^{(2n-1)+(2n-3)+\cdots+3+1}}{(1-q^n)(1-q^{n-1})\cdots(1-q)}\, h_0(q) = \frac{q^{n^2}}{(q;q)_n}\, h_0(q)$$

Since every term in the numerator and denominator of $c(z,q)$ will have a constant factor of $h_0(q)$, we might as well take $h_0(q) = 1$. Thus, $c(z,q)$ can be expressed as

$$c(z,q) = \frac{\sum_{n=0}^{\infty} \frac{q^{n^2}}{(q)_n} z^n}{\sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(q)_n} z^n}$$

and

$$c(1,q) = 1 + \cfrac{q}{1 + \cfrac{q^2}{1 + \cfrac{q^3}{1+\cdots}}} = \frac{G(q)}{H(q)}$$

We have arrived at a celebrated identity of Ramanujan, which would deserve its own paper. We will use this only as a starting point for the study of the series $G(q)$ and $H(q)$. Now we take inspiration from Euler's product formula for the Riemann zeta function. Recall that

$$\zeta(s) = \frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \cdots = \frac{1}{(1-2^{-s})(1-3^{-s})(1-5^{-s})\cdots}$$

where the second equality is informally derived by multiplying $\zeta(s)$ by $(1-2^{-s})$ to cancel out all the multiples of 2, $(1-3^{-s})$ to cancel out all the multiples of 3, and so forth. Our trick is to repeatedly multiply $G(q)$ by terms of the form $(1-q^m)$, where q^m is the smallest nonconstant power of q in the power series expansion of $G(q)$.

$$\begin{aligned}
G(q) &= 1 + \frac{q}{1-q} + \frac{q^4}{(1-q)(1-q^2)} + \frac{q^9}{(1-q)(1-q^2)(1-q^3)} + \cdots \\
(1-q)G(q) &= 1 + \frac{q^4}{1-q^2} + \frac{q^9}{(1-q^2)(1-q^3)} + \frac{q^{16}}{(1-q^2)(1-q^3)(1-q^4)} + \cdots \\
(1-q)(1-q^4)G(q) &= 1 + q^6 + \frac{q^9(1+q^2)}{1-q^3} + \frac{q^{16}(1+q^2)}{(1-q^3)(1-q^4)} + \cdots \\
(1-q)(1-q^4)(1-q^6)G(q) &= 1 + q^9 + q^{11} + q^{14} + \frac{q^{16}(1+q^2)(1+q^3)}{1-q^4} + \cdots
\end{aligned}$$

In view of (2.1), we have $m = 1, 4, 6, 9, 11, \ldots$ for the first few iterations. Notice that m appears to only take on values that are 1 or 4 (mod 5). Hence, we conjecture

$$G(q) \prod_{n=0}^{\infty} (1-q^{5n+1})(1-q^{5n+4}) = 1$$

which is equivalent to (2.2). The reader can check that a similar calculation yields

$$H(q)\prod_{n=0}^{\infty}(1-q^{5n+2})(1-q^{5n+3})=1$$

4. Generating functions of $C_1(n)$ and $C_2(n)$

We derive combinatorial interpretations of the sum side of (2.5) (resp. (2.6)) as generating function of $C_1(n)$ (resp. $C_2(n)$). For all $n \geq 1$, define

$$\mathcal{C}(n)=\{(\lambda_1,\dots,\lambda_k): k\geq 1,\ \lambda_1+\cdots+\lambda_k=n,\ \lambda_i-\lambda_{i+1}\geq 2 \text{ for } i=1,\dots,k-1,\ \lambda_k\geq 1\}$$

to be the set of all partitions of n with no repeated or consecutive parts; in other words, the difference between adjacent parts is at least 2. Similarly, define

$$\mathcal{C}'(n)=\{(\lambda_1,\dots,\lambda_k): k\geq 1,\ \lambda_1+\cdots+\lambda_k=n,\ \lambda_i-\lambda_{i+1}\geq 2 \text{ for } i=1,\dots,k-1,\ \lambda_k\geq 2\}$$

to be the set of all partitions of n with no repeated or consecutive parts and smallest part no less than 2. Thus, $C_1(n)=|\mathcal{C}(n)|$ and $C_2(n)=|\mathcal{C}'(n)|$. We focus on $C_1(n)$, then we outline a similar procedure for $C_2(n)$ The first step is to establish a bijection α between $C_1(n)$ and the set of all partitions of n with smallest part no less than the number of parts; we call this set $\mathcal{B}(n)$. The idea is to take an arbitrary partition in $\mathcal{C}(n)$ and shorten the "gaps" between the parts. For any λ in $\mathcal{C}(n)$, the bijection $\alpha:\mathcal{C}(n)\to\mathcal{B}(n)$ is defined as follows:

$$(\lambda_1,\dots,\lambda_k)\overset{\alpha}{\longleftrightarrow}(\lambda_1-k+1,\dots,\lambda_i+2i-k-1,\dots,\lambda_k+k-1)$$

We see that $\lambda_k+k-1\geq k$, and $(\lambda_i+2i-k-1)-(\lambda_{i+1}+2(i+1)-k-1)\geq 0$ by $\lambda_i-\lambda_{i+1}\geq 2$, for $i=1,\dots,k-1$; the sum of all the parts of the partition is preserved under α by symmetry; thus, α is indeed well-defined.
The inverse $\alpha^{-1}:\mathcal{B}(n)\to\mathcal{C}(n)$ is given by

$$(\lambda_1,\dots,\lambda_k)\overset{\alpha^{-1}}{\longleftrightarrow}(\lambda_1+k-1,\dots,\lambda_i-2i+k+1,\dots,\lambda_k-k+1)$$

and is also well-defined by the definition of $\mathcal{B}(n)$. We conclude that $|\mathcal{C}(n)|=|\mathcal{B}(n)|$, and hence

$$\begin{aligned}\sum_{n=0}^{\infty}C_1(n)q^n&=\sum_{n=0}^{\infty}|\mathcal{B}(n)|q^n\\&=1+\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}|\mathcal{B}(n,k)|q^n\end{aligned}$$

where $\mathcal{B}(n,k)$ is the set of all partitions of n with k parts and $\lambda_k\geq k$. Note that the number of such partitions is the number of partitions of $n-k^2$ into at most k parts: a bijection is given by removing the $k\times k$ Durfee square from the Young Diagram of $\lambda\in\mathcal{B}(n,k)$ (see Figure 4).

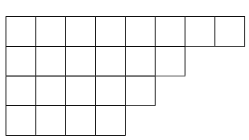


Figure 4. $\lambda=(8,6,5,4)$

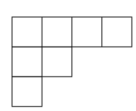


Figure 5. $\lambda=(4,2,1)$

We have

$$\begin{aligned}
1+\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\mathcal{B}(n,k)q^n &= 1+\sum_{k=1}^{\infty}\sum_{n=1}^{\infty}\mathcal{B}(n,k)q^n \\
&= 1+\sum_{k=1}^{\infty}\sum_{n=1}^{\infty}p_k(n-k^2)q^n \\
&= 1+\sum_{k=1}^{\infty}\sum_{n=1}^{\infty}p_k(n)q^{n+k^2} \\
&= 1+\sum_{k=1}^{\infty}q^{k^2}\sum_{n=1}^{\infty}p_k(n)q^n \\
&= 1+\sum_{k=1}^{\infty}\frac{q^{k^2}}{(1-q)(1-q^2)\cdots(1-q^k)} \\
&= G(q)
\end{aligned}$$

Therefore, $G(q)$ is both the generating function for partitions with parts differing by at least two, and the generating function for partitions with smallest part no less than the number of parts.

Let $\mathcal{B}'(n)$ be the set of partitions of n with smallest part strictly greater than the number of parts. The reader can check that $\mathcal{C}'(n)$ is isomorphic to $\mathcal{B}'(n)$ by

$$(\lambda_1,\ldots,\lambda_k)\longleftrightarrow(\lambda_1-k+1,\ldots,\lambda_i+2i-k-1,\ldots,\lambda_k+k-1)$$

and that $\mathcal{B}'(n)$ is in turn isomorphic to the set of all partitions of $n-k^2-k$ with at most k parts (Figure 6 gives an explicit bijection). Putting this together, we have

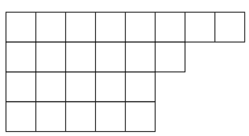


Figure 6. $\lambda=(8,6,5,5)$

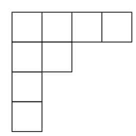


Figure 7. $\lambda=(4,2,1,1)$

$$\begin{aligned}
\sum_{n=0}^{\infty}C_2(n)q^n=\sum_{n=0}^{\infty}|\mathcal{B}'(n)|q^n &= 1+\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}|\mathcal{B}'(n,k)|q^n \\
&= 1+\sum_{k=1}^{\infty}\sum_{n=1}^{\infty}p_k(n-k^2-k)q^n \\
&= 1+\sum_{k=1}^{\infty}\sum_{n=1}^{\infty}p_k(n)q^{n+k^2+k} \\
&= 1+\sum_{k=1}^{\infty}\frac{q^{k^2+k}}{(1-q)(1-q^2)\cdots(1-q^k)} \\
&= H(q)
\end{aligned}$$

where $\mathcal{B}'(n,k)$ is defined similarly as $\mathcal{B}(n,k)$.

5. Combinatorial proof

In this section, we will state and prove the two symmetry equations, ultimately leading to a proof of Theorem 2.3. One of the main ideas (due to Andrews [And79]) is to use iterated Durfee squares, or more generally, Durfee m-rectangles, to study the Rogers Ramanujan identities.

By construction, the Durfee m-rectangle splits a partition λ into two smaller partitions to the right of and below the rectangle. As long as the λ is not in Q, the lower partition will be nonempty. Therefore, we can further separate the lower partition into two smaller partitions via its Durfee m-rectangle (see Figure). Thus, λ consists of two iterated Durfee m-rectangles, a partition α to the right of the larger rectangle, a partition β between the two rectangles, and a partition γ below the smaller rectangle. Let $s_m(\lambda)$ (resp. $t_m(\lambda)$ denote the height of the first (resp. second) iterated Durfee m-rectangle; we write $s = s_m(\lambda)$ and $t = t_m(\lambda)$ for short. By construction, we have $s \geq t$. We are now ready to define the $(2, m)$-rank of a partition.

Definition 5.1. For $m \geq 1$, or $m = 0$ and $\lambda \in P \setminus Q$, the $(2, m)$-*rank* of λ $r_{2,m}(\lambda)$ is given by

$$r_{2,m}(\lambda) = \beta_1 + \alpha_{s-t-\beta_1+1} - \gamma'_1$$

The statistic $r_{2,m}(\lambda)$ is a special case of the more general (k, m)-rank, used by Andrews [And79] to prove his generalizations of the Rogers Ramanujan identities.

We can now define sets of partitions based on the notion of $(2, m)$-rank. Let $\mathcal{H}_{n,m,r}$ denote set of partitions of n with $(2, m)$-rank equal to r. Similarly define $\mathcal{H}_{n,m,\leq r}$ and $\mathcal{H}_{n,m,\geq r}$ in the obvious fashion. As for the sizes of these sets, let $h(n, m, r) = |\mathcal{H}(n, m, r)|$, $h(n, m, \leq r) = |\mathcal{H}(n, m, \leq r)|$, and $h(n, m, \geq r) = |H(n, m, \geq r)|$. Clearly, all partitions of n have $(2, m)$-rank either less than r or no less than r; this implies,

$$p(n) = h(n, m, \leq r) + h(n, m, \geq r + 1)$$

for all $r \in \mathbb{Z}$ and $m > 0$. If $m = 0$, we have

$$p(n) - q(n) = h(n, 0, \leq r) + h(n, 0, \geq r + 1)$$

The proofs of the following two symmetries will rely solely on combinatorial methods; we will save their algebraic relevance for the next section.

$$h(n, 0, r) = h(n, 0, -r) \qquad \text{(first symmetry)}$$
$$h(n, m, \leq -r) = h(n - r - 2m - 2, m + 2, \geq -r) \qquad \text{(second symmetry)}$$

5.1. **Proof of the first symmetry.** The idea is to define a two-step involution φ on $P \setminus Q$ which preserves the Durfee squares but changes the sign of the rank. The map φ can be thought of as sending (α, β, γ) to some 5-tuple of partitions $(\mu, \nu, \pi, \rho, \sigma)$, and then to a new triple $(\hat{\alpha}, \hat{\beta}, \hat{\gamma})$. For any $\lambda \in P \setminus Q$, its image $\varphi(\lambda)$ is constructed by deleting α, β, γ and replacing them with $\hat{\alpha}, \hat{\beta}, \hat{\gamma}$, respectively.

Definition 5.2. (Definition of φ)

Step 1: Let $\mu = \beta$.

Remove the parts $\alpha_{s-t-\beta_j+j}$ for $1 \leq j \leq t$; let ν consist of the parts removed, and π consist of the parts not removed from α (assume all the parts are arranged in nonincreasing order so that ν and π are well-defined).

For each $1 \leq j \leq t$, define

$$k_j = \max\{k \leq s - t : \gamma'_j - k \geq \pi_{s-t-k+1}\}$$

For $1 \leq j \leq t$, define $\rho_j = k_j$ and $\sigma_j = \gamma'_j - k_j$.

Step 2: Define $\hat{\alpha} = \pi \cup \sigma$, $\hat{\beta} = \rho$, and $\hat{\gamma}' = \mu + \nu$ (see Figure **??** for an example).

It turns out to be more convenient to talk about things in terms of the conjugate of γ rather than the partition γ itself; since γ' is attached to the second Durfee square (which has width $t - m$), we have $\ell(\gamma') \leq t - m$. Thus, γ' has a fixed upper bound on the number of parts, unlike γ.

Let us elaborate on the definition of k_j. Define the set $S_j = \{k \leq s - t : \gamma'_j - k \geq \pi_{s-t-k+1}\}$ for convenience. Note that 0 is always in S_j because $\pi_{s-t+1} = 0$, so k_j is indeed well-defined. Since

$k_j = \max S_j$, we have $k_j + 1 \notin S_j$. If $k_j < s - t$, then $\gamma'_j - (k_j + 1) < \pi_{s-t-(k_j+1)+1}$, so we have $\gamma'_j - k_j \leq \pi_{s-t-k_j}$. This gives the useful inequality

$$\pi_{s-t-k_j+1} \leq \gamma'_j - k_j \leq \pi_{s-t-k_j} \tag{5.1}$$

as long as $k_j < s - t$ (we consider the case when $k_j = s - t$ separately). In fact, we chose k_j to be the unique $k < s - t$ that satisfies the inequality (5.1). To see this, suppose we have $k \leq s - t$ and $\pi_{s-t-k+1} \leq \gamma'_j - k \leq \pi_{s-t-k}$. This implies k is in S_j, while $k + 1$ is not. But note whenever we have k_1, k_2 such that $k_1 \in S_j$ and $k_2 < k_1$, k_2 must also be in S_j by the inequality $\gamma'_j - k_2 > \gamma'_j - k_1 \geq \pi_{s-t-k_1+1} \geq \pi_{s-t-k_2+1}$. Thus, k can only be the maximum of the set S_j.

We now begin the core of the proof of the first symmetry.

Lemma 5.3. *The map $\varphi : P \setminus Q \to P \setminus Q$ is well-defined.*

Proof. Let us check that ρ and σ are indeed partitions, i.e. $\rho_1 \geq \rho_2 \geq \cdots \geq \rho_t$ and $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_t$. We first prove $k_j \geq k_{j+1}$ for $j = 1, \ldots, t-1$. Informally, the condition $\gamma'_{j+1} - k \geq \pi_{s-t-k+1}$ is "stricter" than the condition $\gamma'_j - k \geq \pi_{s-t-k+1}$ since $\gamma'_{j+1} \leq \gamma'_j$. In other words, we have $k_{j+1} + \pi_{s-t-k_{j+1}+1} \leq \gamma'_{j+1} \leq \gamma'_j$; thus, $k_{j+1} \in S_j$ and $k_{j+1} \leq \max S_j$. Next, we verify that $\gamma'_j - k_j \geq \gamma'_{j+1} - k_{j+1}$; this is clear if $k_j = k_{j+1}$. Otherwise, $k_{j+1} \leq k_j - 1$ and $k_{j+1} < s - t$, so

$$\pi_{s-t-k_{j+1}+1} \leq \gamma'_{j+1} - k_{j+1} \leq \pi_{s-t-k_{j+1}} \leq \pi_{s-t-k_j+1} \leq \gamma'_j - k_j$$

Now it suffices to show that $\hat{\alpha}$ has at most s parts, $\hat{\beta}$ has at most t parts with largest part no larger than $s - t$, and $\hat{\gamma}'$ has at most t parts. The first claim follows by the definition $\hat{\alpha} = \pi \cup \sigma$, π having at most $s - t$ parts and σ having at most t parts. Next, $\hat{\beta} = \rho$ has at most t parts by definition and $\hat{\beta}_j = k_j \leq s - t$ for all $1 \leq j \leq t$. Finally, μ and ν each have at most t parts, so their sum $\hat{\gamma}'$ has at most t parts as well.

It is now clear that the first and second Durfee squares of $\varphi(\lambda)$ are indeed the same as the ones from λ, and since the second square is well-defined, $\varphi(\lambda)$ is indeed a non-Rogers-Ramanujan partition. ■

Let us introduce some notation for Lemma 5.4, which states that φ^2 is the identity map. We will apply φ to $\lambda \in P \setminus Q$ twice and show that $\varphi^2(\lambda) = \lambda$. For $1 \leq j \leq t$, we define

$$\hat{k}_j = \max\{k \leq s - t : \hat{\gamma}'_j - k \geq \pi_{s-\hat{t-k}+1}\}$$

Let $(\hat{\mu}, \hat{\nu}, \hat{\pi}, \hat{\rho}, \hat{\sigma})$ be the 5-tuple of partitions given by applying the first step of φ to $\varphi(\lambda)$, and let $(\alpha^*, \beta^*, \gamma^*)$ be the triple obtained from the second step. Thus, Lemma 5.4 is equivalent to showing that $\alpha = \alpha^*$, $\beta = \beta^*$, and $\gamma = \gamma^*$. As the notation can become quite cumbersome, refer to Figure 8.

Lemma 5.4. *The map φ is an involution.*

Proof. First, observe that $\rho = \hat{\beta} = \hat{\mu}$ by definition.

We know from (5.1) that $\pi_{s-t-k_j+1} \leq \sigma_j \leq \pi_{s-t-k_j}$, assuming $k_j < s - t$. In other words, σ_j is "sandwiched" between two consecutive parts of π; this tells us how the parts of $\pi \cup \sigma$ are ordered:

$$\sigma_{\ell(\sigma)}, \ldots, \sigma_{j+1}, \pi_{\ell(\pi)}, \ldots, \pi_{s-t-k_j+1} \leq \sigma_j \leq \pi_{s-t-k_j}, \ldots, \pi_1, \sigma_{j-1}, \ldots, \sigma_1$$

Since there are $s - t - k_j + j$ terms on the right hand side, we conclude that $\sigma_j = \hat{\alpha}_{s-t-k_j+j}$. Therefore, $\sigma_j = \hat{\nu}_j$; of course, this is true even if $k_j = s - t$.

Thus, we have $\gamma' = \sigma + \rho = \hat{\nu} + \hat{\mu} = (\gamma^*)'$.

Next, $\pi \cup \sigma = \hat{\alpha} = \hat{\pi} \cup \hat{\nu}$ and $\sigma = \hat{\nu}$ implies that $\pi = \hat{\pi}$.

If $\hat{k}_j < s - t$, then $\hat{k}_j$ is the unique integer at most $s - t$ satisfying

$$\hat{\pi}_{s-t-\hat{k}_j+1} \leq \hat{\gamma}'_j - \hat{k}_j \leq \hat{\pi}_{s-t-\hat{k}_j}$$

but since $\hat{\pi} = \pi$ and $\hat{\gamma}'_j = \mu_j + \nu_j = \beta_j + \nu_j$, we have

$$\pi_{s-t-\hat{k}_j+1} \leq \beta_j + \nu_j - \hat{k}_j \leq \pi_{s-t-\hat{k}_j}$$

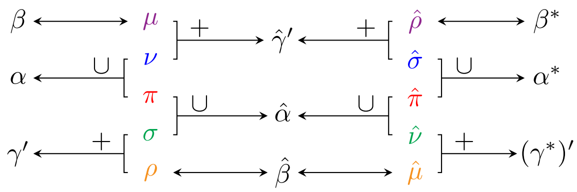


Figure 8. Visual accompaniment to Lemma 5.4. Equal partitions are highlighted in the same color. Arrows going in one direction represent union and addition; arrows going in both directions signify equality.

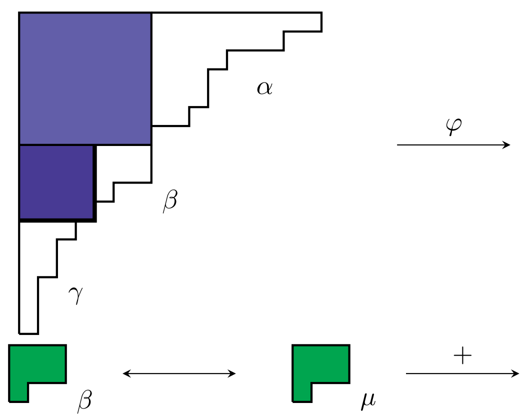
By definition, $\nu_j = \alpha_{s-t-\beta_j+j}$. Consider the parts $\alpha_1, \alpha_2, \ldots, \alpha_{s-t-\beta_j+j-1}$. Exactly $j-1$ of these are parts of ν, namely $\nu_1, \ldots, \nu_{j-1}$; the remaining $s-t-\beta_j$ parts are parts of π, namely $\pi_1, \ldots, \pi_{s-t-\beta_j}$. We conclude that

$$\pi_{s-t-\beta_j+1} \leq \nu_j \leq \pi_{s-t-\beta_j}$$

and

$$\pi_{s-t-\beta_j+1} \leq \beta_j + \nu_j - \beta_j \leq \pi_{s-t-\beta_j}$$

so $\mu_j = \beta_j = \hat{k}_j = \hat{\rho}_j$ where the middle equality is by the uniqueness of $\hat{k}_j$. It follows that $\beta = \beta^*$. Finally, we have $\mu + \nu = \hat{\gamma}' = \hat{\rho} + \hat{\sigma}$ and $\mu = \hat{\rho}$, so $\nu = \hat{\sigma}$. Therefore, $\alpha = \nu \cup \pi = \hat{\sigma} \cup \hat{\pi} = \alpha^*$. ■



Lemma 5.5. *If $\lambda \in P \setminus Q$ has $(2, m)$-rank $r_{2,m}(\lambda) = r$, then $r_{2,m}(\varphi(\lambda)) = -r$.*

Proof. Suppose $\beta_1 + \alpha_{s-t-\beta_1+1} - \gamma_1' = r$. In order to prove that $\hat{\beta}_1 + \hat{\alpha}_{s-t-\hat{\beta}_1+1} - \hat{\gamma}_1' = -r$, we can rewrite everything in terms of μ, ν, π, ρ, and σ.

First, note that $\beta_1 + \alpha_{s-t-\beta_1+1} - \gamma_1' = \mu_1 + \nu_1 - \rho_1 - \sigma_1$. Using the result that $\hat{\nu} = \sigma$ from Lemma 5.4, we have $\hat{\alpha}_{s-t-\hat{\beta}_1+1} = \hat{\nu}_1 = \sigma_1$. Furthermore, $\hat{\beta}_1 = \rho_1$ and $\hat{\gamma}_1' = \mu_1 + \nu_1$. Hence, $\hat{\beta}_1 + \hat{\alpha}_{s-t-\hat{\beta}_1+1} - \hat{\gamma}_1' = \rho_1 + \sigma_1 - \mu_1 - \nu_1$. ■

5.2. **Proof of the second symmetry.** We wish to establish a bijection

$$\psi_{m,r} : \mathcal{H}_{n,m,\leq -r} \to \mathcal{H}_{n-r-2m-2,m+2,\geq -r}.$$

For reasons that will become apparent, we require the two Durfee m-rectangles to have nonzero width. If $m = 0$, then the heights being nonzero guarantees nonzero widths. If $m > 0$, note that the second Durfee m-rectangle has nonzero width if $\gamma_1' > 0$; by extension, the first m-rectangle also has nonzero width.

Definition 5.6. (Definition of $\psi_{m,r}$)

6. Proof of Schur's Identity

For $j \geq 0$, we define

$$a_j = h(n - jr - 2jm - j(5j-1)/2, m + 2j, \leq -r - j)$$
$$b_j = h(n - jr - 2jm - j(5j-1)/2, m + 2j, \geq -r - j + 1)$$

By the second symmetry, we have

$$b_{j+1} = h(n - jr - 2jm - j(5j-1)/2 - r - 2m - 5j - 2, m + 2j + 2, \geq -r - j) = a_j$$

The point is to start with $a_0 = h(n, m, \leq -r)$ and repeatedly add zero in the form of $(a_j - b_{j+1})$ or $-(a_j - b_{j+1})$. Thus,

7. Second Rogers-Ramanujan identity

8. Final remarks

References

[AB89] George E Andrews and Rodney J Baxter. A motivated proof of the rogers-ramanujan identities. *The American mathematical monthly*, 96(5):401–409, 1989.

[And79] George E. Andrews. Partitions and durfee dissection. *American Journal of Mathematics*, 101(3):735–742, June 1979.

[And89] George E Andrews. On the proofs of the rogers-ramanujan identities. In *q-Series and Partitions*, pages 1–14. Springer, 1989.

[Dys44] Freeman J Dyson. Some guesses in the theory of partitions. *Eureka (Cambridge)*, 8(10):10–15, 1944.

[Pak06] Igor Pak. Partition bijections, a survey. *The Ramanujan Journal*, 12(1):5–75, August 2006.

[SF82] James Joseph Sylvester and Fabian Franklin. A constructive theory of partitions, arranged in three acts, an interact and an exodion. *American Journal of Mathematics*, 5(1):251–330, 1882.