Euler-Maclaurin Formula

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Background

Euler discovered the Euler-Maclaurin Formula in 1732 through the Basel Problem, which asks for the value of $\sum_{n=1}^{\infty} \frac{1}{n^2}$. It was later independently discovered by Maclaurin in 1742. The Basel problem had stumped mathematicians for around 90 years prior to Euler's solution.

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Euler's solution consisted of solving for the coefficient of x^3 in the infinite expansion of $\sin x$ in two ways. He calculated it as $-\frac{1}{6}$ through the Maclaurin Series of $\sin x$ and used a factorization known as the Weirstrass Factorization to show that the coefficient could also be expressed as $-\sum_{n=1}^{\infty} \frac{1}{(n\pi)^2}$. Setting these equal with some slight rearranging gives us that the answer to the Basel Problem is $\frac{\pi^2}{6}$

Background

Euler seemed to be dissatisfied with only an answer to the Basel Problem and wanted to be able to solve or approximate infinite series with their respective integrals. However, when deriving the Euler-Maclaurin Formula, both Euler and Maclaurin were unable to solve for the exact remainder term and it wasn't until 1823 when Poisson discovered it.

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The first few terms are as follows:

n	b_n
0	1
1	-$\frac{1}{2}$
2	$\frac{1}{6}$
3	0
4	$-\frac{1}{30}$
5	0
6	$\frac{1}{42}$

Definitions

There happens to be no simple pattern to the Bernoulli numbers, but a good approximation is

$$b_{2n} \approx (-1)^{n-1} 4\sqrt{\pi n} (\frac{n}{\pi e})^{2n}$$

for large values of n.

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In a similar fashion, we can define the Bernoulli Polynomial for nonnegative integers n as such:

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The first few polynomials can be expressed as follows:

$$B_0(x) = 1$$

$$B_1(x) = x - \frac{1}{2}$$

$$B_2(x) = x^2 - x + \frac{1}{6}$$

Definitions

Definition

We define the Periodic Bernoulli Function as such, where $\{x\} = x - \lfloor x \rfloor$:

$$P_n(x) = B_n(\{x\})$$

This will be used to express the remainder term.

Euler-Maclaurin Formula

Euler-Maclaurin theorem

For a function $f(x)$ that is p times differentiable on the interval $[m, n]$ for a positive integer p, we have

$$\sum_{i=m}^{n} f(i) = \int_m^n f(x)dx + \frac{f(n)+f(m)}{2} + \sum_{k=2}^{p} \frac{b_k}{k!}\left(f^{(k-1)}(n) - f^{(k-1)}(m)\right) + R_p$$

where

$$R_p = (-1)^{p+1} \int_m^n f^{(p)}(x) \frac{P_p(x)}{p!} dx$$

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$$du = f'(x)dx$$
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$$\int_k^{k+1} f(x)dx = [uv]_k^{k+1} - \int_k^{k+1} vdu$$

$$= B_1(1)f(k+1) - B_1(0)f(k) - \int_k^{k+1} f'(x)P_1(x)dx$$

Proof

Summing k from 0 to $n-1$ and plugging in $B_1(0) = -\frac{1}{2}, B_1(1) = \frac{1}{2}$, we get

$$\int_0^1 f(x)dx + \int_1^2 f(x)dx + \dots \int_{n-1}^n f(x)dx = \int_0^n f(x)dx$$

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$$\int_0^1 f(x)dx + \int_1^2 f(x)dx + \dots \int_{n-1}^n f(x)dx = \int_0^n f(x)dx$$

$$= \frac{f(0)}{2} + f(1) + f(2) + \dots + f(n-1) + \frac{f(n)}{2} - \int_0^n f'(x)P_1(x)dx$$

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Some simple rearranging gives us

$$\sum_{k=0}^{n} f(k) = \int_0^n f(x)dx + \frac{f(n) + f(0)}{2} + \int_0^n f'(x)P_1(x)$$

which concludes the base case.

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Consider the error term for the case $p = 1$

$$\int_k^{k+1} f'(x)P_1(x)dx = \int_k^{k+1} udv$$

where

$$u = f'(x)$$

$$du = f''(x)dx$$

$$dv = P_1(x)dx$$

$$v = \frac{1}{2}P_2(x)$$

Proof

Again, integrating by parts gives us

$$\int_k^{k+1} f'(x)P_1(x)dx = [uv]_k^{k+1} - \int_k^{k+1} vdu$$

$$= \frac{b_2}{2}(f'(k+1) - f'(k)) - \frac{1}{2}\int_k^{k+1} f''(x)P_2(x)dx$$

Summing k from 0 to $n-1$ once again gives us the $p = 2$ case where

$$\sum_{k=0}^{n} f(k) = \int_0^n f(x)dx + \frac{f(n)+f(0)}{2} + \frac{b_2}{2}(f'(k+1) - f'(k)) - \frac{1}{2}\int_0^n f''(x)P_2(x)$$

Euler's Constant

Something interesting happens when we consider the formula for $f(x) = \frac{1}{x}$. Let $m = 1$ and $p = 1$ in the formula and we result the following:

$$\sum_{i=1}^{n} \frac{1}{i} = \log n + \frac{1}{2n} + \frac{1}{2} + \int_1^n \frac{P_1(x)}{x^2} dx$$

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Note that $P_1(x)$'s absolute value is bounded by $\frac{1}{2}$ and thus $R(n)$ converges when n approaches ∞. Let us denote $\gamma = \lim_{n \to \infty} R(n)$.

Euler's Constant

A good approximation for Euler's Constant is $\gamma = 0.57721566490153286060651209 00\ldots$. It's still an open problem whether γ is rational or not and some even believe it is transcendental. What do you think it is?