Permutations in Analytic Combinatorics

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What is combinatorics?

The main problem in combinatorics is finding a finite set of mathematical objects in the limits of certain constraints such as those in graph theory. Combinatorics concerns itself with counting as a means of obtaining a result and looks at properties of finite structures. These principles of counting are fundamental to topics such as probability and build a foundation for various mathematical areas. Enumeration is much like a foundation for a house, a simple yet necessary part of construction.

What is analytic combinatorics?

Use of complex analysis and generating functions to enumerate and understand the growth rates of combinatorial objects. It has proven to be a great tool for predicting precise properties of large combinatorial structures that can be used for the *analysis of algorithms*.

Example: *Quicksort*:

1. Pivot Element 2. Arrays 3. Lesser than pivot element 4. Greater than pivot element

In other words: a great book recommendation!

To analyse algorithms

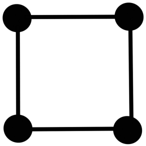
Symbolic Method: *to derive a direct Generating Function for a dataset*

One can do this by defining a class of combinatorial objects.

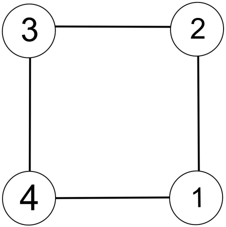
Combinatorial Classes

1. Unlabelled Structures
2. Labelled Structures
   Suppose objects composed of N atoms in that each atom is labelled by integers 1 through n. In *Labelled Structures* each atom is distinctly labelled and cannot correspond with another atom even in similar combinatorial structures unlike unlabelled classes in which atoms are distinct however are not distinctly labelled such that similar unlabelled combinatorial structures are not corresponding.

Unlabelled Structures



Labelled Structures



What are generating functions?

Generating functions provide a systematic manner of enumerating combinatorial objects as well as a way of discovering the properties of these objects that make up large combinatorial structures used within algorithms.
The use of generating functions helps find sequences for finite numbers in a large combinatorial structure which helps with enumeration. Instead of an infinite sequence, generating functions encode these sequences as a single function of a formal power series like those in calculus. Therefore, the use of generating functions produces a power series that helps keep track of numbers in an infinite sequence and represent a *generating series* of numbers.

Ordinary Generating Functions

Using the example of a binary number sequence, we suppose a set A where

$$A = \varepsilon, 0, 1, 00, 01, 10, 11, 000, 001, 010, \ldots$$

Each letter has two possibilities and each possibility multiplies which dictates that the cardinality of set A will be written as

$$A_n = 2^n$$

This depicts an *unlabelled combinatorial class* which would then be represented by an ordinary generating function as a formal power series

$$A(z) = \sum_{n=0}^{\infty} A_n z^n = A(z) = \sum_{a \in A} z^{|a|}$$

Exponential Generating Functions

Using the same set A, we may suppose all numbers within the set are distinct from one another so that no set or depicted graph is identical to A. This would mean that A is a *labelled combinatorial class* and can therefore be represented by an *exponential generating function*:

$$A(z) = \sum_{n \geq 0} A_n \frac{z^n}{n!} = \sum_{a \epsilon A} \frac{z^{|a|}}{|a|!}$$

An example of using combinatorics

The result of sequences by Generating Functions can be used to calculate permutations (referring to the arrangement of a set of numbers in a specific order).
If one were to use n amount of distinct objects in a sequence, an ordered arrangement of these objects could make a permutation. Therefore, a permutation of n would be found by $n!$ as it would provide us with the product of all integers from 1 to n.

Permutations

iPhones have 4-digit passcodes. If we suppose that there are 4 smudges over 4 different numbers on the screen. How many distinct passcodes would be possible?

Solution

1. Order?
2. Distinct?
3. Permutation or Combination?

Solution:

$$4! = 24$$

Permutations in Combinatorics?

Combinations: choosing winners in a team
Permutations: selecting winners by their positions i.e first, second and third
EXAMPLE: Distinct ways of ordering letters in the word MISSISSIPPI
Combinatorial word structure:

$$A = (M, I, I, I, I, S, S, S, S, P, P)$$

Supposing all letters are distinguishable irrespective of repeating, we must solve the problem of overcounting by using permutations. Therefore, instead of 11!, our end result would be

$$\frac{11!}{4!.4!.2!} = 34,650$$

Permutation Statistics

Random Permutation statistics are oftentimes vital to the *analysis of algorithms* especially in sorting algorithms such as **Quicksort** which operates on the foundation of random permutations. Let us take the example of *Quickselect* which selects particular data from a random permutation. Naturally, when large sets of data are selected or sorted through, they are disordered. The disordered data can be analysed by using generating functions that dictate a sequence of these large data sets by using generating functions produced of random permutations.

Erdős-Szekeres Theorem

Statement: Every infinite sequence of real numbers which are distinct produces a monotonically increasing or decreasing infinite subsequence.
Given r and s, any sequence of distinct real numbers with length at least

$$(r-1)(s-1)+1$$

contains a monotonically increasing or decreasing infinite subsequence. Therefore, if r=3 and s=2, the formula shows that any permutations of three numbers would have an increasing subsequence of 3 and a decreasing subsequence of 2.

Young Tableau and permutations

A Young Tableau is a set written within a finite number of boxes in a grid-like pattern oftentimes presenting data in an *ordered set*, therefore, this produces ordered sequences much like those represented through permutations

Conclusion

Contents of the paper:
-Introduction to generating functions, use of permutations in analytic combinatorics and the division of combinatorial classes
-Application of permutations in analytic combinatorics and proofs
-The use of permutations and the effectiveness of permutation statistics
-The use of generating functions resulting in permutations and its application to limit shapes.