Random Permutations using Analytic Combinatorics

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June 2022

1 Introduction

This paper focuses on studying the characteristics of analytic combinatorics, specifically combinatorial classes, labelled and unlabelled structures, and the specific combinatorial class of permutations. The use of permutations within analytic combinatorics is defined through generating functions which are explored through examples so that there is ample understanding of the workings of sequences of infinite and finite sets of permutations. The paper dives into the combinatorial class of permutation through the functionings of analytic combinatorics and explores its applicability to limit shapes through specific examples of permutations. This paper specifically explores permutations within limit shapes through the Erdos-Szekeres theorem and the square Young tableau.

2 Combinatorial Classes

Definition 2.1. Combinatorics principally deals with finding a finite set of mathematical objects filtered by a parameter n. These finite sets are used to plot each object from within the set to a *nonnegative integer* such that there are finitely many objects of each size. These countable sets of mathematical objects are referred to as *combinatorial classes.*

Conditions 2.2. A key question in combinatorics is to enumerate these objects under their set parameters:

1. *the size of an element is a nonnegative integer*
2. *the number of elements of any size are finite*

2.1 Unlabelled Classes

Definition 2.3. Supposing a combinatorial class A in which $a \in A$, the size of a will be written as $|a|$. In a few special cases where further specification is required, the size will be written as $|a|_A$ when the same element is used for different sizes in different combinatorial classes. The subset of A containing the elements of size n will be written as A_n whereas the cardinality of subset A_n will be written as a_n.

While combinatorial problems vary in difficulty, a basic problem is that of finding binary sequences of elements within a set. A simple example is that of binary sequences using the set A with elements taken from the binary alphabet.

Example 1. Our combinatorial class would then be:

$$A = [E, 0, 1, 00, 01, 10, 11, 000, 001, 010, 011, 100, 101, 110, 111, ...].$$

in that E represents an empty sequence. Each letter has 2 possibilities and all possibilities multiply, thus, letting the cardinality of S_n be s_n, it is apparent that $s_n = 2^n$.

Another important example is that of *permutations* which will be discussed further in 4.

Example 2. A permutation of size n has one-to-one correspondence with the integer interval derived from computer science:

$$I_n := [1..n]$$

Therefore, it can be represented as an array:

$$\begin{matrix} 1 & 2 & \dots & n \\ \sigma_1 & \sigma_2 & \dots & \sigma_n \end{matrix}.$$

It can also be represented as a sequence of its definite elements: $\sigma_1\sigma_2\dots\sigma_n$ For the set S:

$$S = [\dots, 12, 21, 123, 132, 213, 312, 321, 1234, \dots, 532614, \dots].$$

Consider a permutation sequence consisting of n definite numbers in which there are n places for the placement of n, $n-1$ places for $n-1$, $n-2$ places for $n-2$, and so forth. Thus, the permutations for size n of P_n would satisfy:

$$P_n = n! = 1.2.\dots.n..$$

Another combinatorial class is that of binary trees which consists of empty nodes or a root node. Each node then extends forward to the left or right of their position. The size of each tree is determined by the numbers of its contained nodes, however, it can be determined by other factors such as its height or its leaves. While counting binary trees is a bit more challenging than sequences or permutations as were shown in 1 and 2 respectively, the number of binary trees of size n is equivalent to the *nth Catalan number* $C_n = \frac{1}{n+1}\binom{2n}{n}$.

Catalan numbers are also used to enumerate numerous other combinatorial classes that will be discussed further. Interestingly, these classes are all combinatorically the same outside of their superficial differences. Combinatorial classes similar to binary trees include parenthesized $n+1$ symbols, regular $(n+2)$-gons and various others. These combinatorial classes, therefore, become *isomorphic* to one another.

Example 3. Suppose combinatorial class $\mathcal{A}$ and $\mathcal{B}$ are isomorphic and write $A \cong B$ if $a_n = b_n$ for all $n \geq 0$

Unlabelled Classes are translated over *Ordinary Generating Functions* and so the ordinary generating function of the combinatorial class $\mathcal{A}$:

$$A(z) = \sum_{n=0}^{\infty} A_n z^n$$

translates to the generating function of the numbers A_n:

$$A(z) = \sum_{a \in A} z^{|a|}.$$

The combinatorial form in this case occurs when supposing the z^n occurs as many times as size n appears for the objects in $\mathcal{A}$. Principally, generating functions are considered a **formal power** series in this case. Generating functions are discussed more in depth in 3.

2.2 Labelled Classes

Definition 2.4. A *labelled structure* in a combinatorial class consists of distinctly labelled atoms making up an object. For instance, instead of using a class of graphs we may use a class of labelled graphs in which all vertices n are distinguished by a distinct label from 1 to n so that no graph is identical besides in the case of identically labelled correspondence of nodes between two graphs.

In order to count labelled functions, we turn to *exponential generating functions (EGF)*

Example 4. The exponential generating function (EGF) of series $\mathcal{A}_n$ is the formal power series:

$$A_n = \sum_{n \geq 0} A_n \frac{z^n}{n!}$$

The exponential generating function of class $\mathcal{A}$ would become the exponential generating function of the numbers of A_n and therefore would present the function as:

$$A(z) = \sum_{n \geq 0} A_n \frac{z^n}{n!} = \sum_{a \epsilon A} \frac{z^{|a|}}{|a|!}.$$

where the variable z marks the size in the generating function.

For labelled enumeration, the most important examples prove to be those of **circles, urns and permutations** shown in 5 till 7

Example 5. *Permutations.* For class $\mathcal{P}$, under the linear representation of permutations where:

$$\begin{array}{cccc} 1 & 2 & \dots & n \\ \sigma_1 & \sigma_2 & \dots & \sigma_n \end{array}$$

can also be presented as a sequence

$$\sigma_1, \sigma_2, \ldots, \sigma_n.$$

Class $\mathcal{P}$ can therefore, be presented as

$$\mathcal{P} = \left\{ \epsilon\ ,\ ①,\ \begin{array}{c} ①-② \\ ②-① \end{array},\ \begin{array}{c} ①-②-③ \\ ②-③-① \\ ③-①-② \\ ②-①-③ \\ ①-③-② \\ ③-②-① \end{array},\ \ldots \right\},$$

such that

$$P_0 = 1, P_1 = 1, P_2 = 2, P_3 = 6.$$

This produces the class $\mathcal{P}$ of a labelled enumeration in the form of the sequence defined by

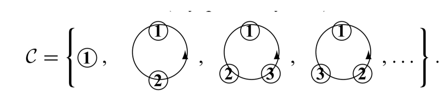
$$P_n = n!$$

since it would follow the same rule established in 2 stating that there would be n choices for the placement of element 1, $n-1$ for element 2, so on and so forth. This would produce the *Exponential Generating Function* of $\mathcal{P}$ as

$$P(z) = \sum_{n \geq 0} n! = \sum_{n \geq 0} z^n = \frac{1}{1-z}.$$

Permutations within combinatorics provide essential due to their characteristic of ordering elements proving to be useful within order statistics.

Example 6. *Circular Graphs.* The class $\mathcal{C}$ of *circular graphs* is bijective to *cyclic permutations* and proves to be an important example to consider within this paper. In this class, cycles are oriented in a positively conventional manner:

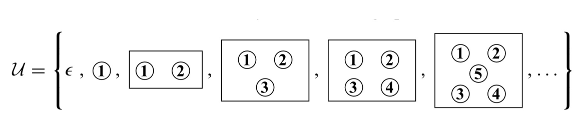


Since a directed cycle is based on its succession of elements following 1, one has $C_n = (n-1)!$ presented as an EGF:

$$C(z) = \sum_{n \geq 1} (n-1)! \frac{z^n}{n} = log \frac{1}{1-z}.$$

The logarithm is rather characteristic for circular arrangements within labelled objects.

Example 7. *Urns.* The class $\mathcal{U}$ is shown as disconnected graphs:



To specify, the order of the labelled atoms does not matter in the class of *urns* such that for each n, there is only a single possible arrangement and $U_n = 1$. The class $\mathcal{U}$ contains urns of size n which contain n distinguishable balls in an undefined order. The EGF for class $\mathcal{U}$ would then become:

$$U(z) = \sum_{n \geq 0} 1 \frac{z^n}{n!} = exp(z) = e^z.$$

While urns look trivial, they provide as building blocks for complex labelled structures such as allocations within sorting. The purpose of this paper, however, focuses on the use of permutations derived from analytic combinatorics and will primarily place emphasis on the details of permutations.

3 Generating Functions

Definition 3.1. A Generating Function represents the recurrence of a function within an infinite *generating sequence*. It is presented as a formal power series to define an infinite sequence and helps keep track of the nth term within an infinite dataset.

The classification of generating functions into ordinary generating function (OGF) and Exponential Generating Functions (EGF) provides a distinction between unlabelled and labelled combinatorial classes as seen in 2

3.1 Ordinary Generating Functions

Definition 3.2. The Ordinary Generating Function of a sequence represents a recursive linear sequence consisting of constant coefficients. Such generating functions are seen in unlabelled combinatorial classes where sequences are recurring and linear.

Example 8. Let us consider an example with subsets where the size of subsets is equivalent to the number of elements in the subset. Suppose $\}$ is a positive integer.

The generating function of the sequence $a_n = \binom{k}{n}$ for $n < k$ is a polynomial and is presented as:

$$A(x) = \sum_{n \geq 0} \binom{k}{n} x^n = (1 + x)^k.$$

Using the second equality, we use the binomial theorem and consider the sequence represented as a generating function of the subset $(1, 2, \ldots, k)$.

Using this example, let us suppose the probability g of flipping a coin and landing on heads.The probability of the coin landing on tails would be represented by $p = 1 - g$. We toss it k times and use the defined cardinality of a^n as with all combinatorial sequences to denote the probability of getting heads

n times. Using the binomial theorem again, we see that $a_n = \binom{k}{n} q^{k-n} p^n$ and so represents the generating function:

$$A(x) = \sum_{n \geq 0} \binom{k}{n} q^{k-n} p^n x^n = (q + px)^k,$$

which can be represented as

$$(q + px)(q + px) \ldots (q + px).$$

This shows a recurring sequence represented by an ordinary generating function.

Example 9. Suppose a fibonacci sequence:

$$1 + z + 2z^2 + 3z^3 + 5z^4 + 8z^5 + \ldots = \sum_{n=0}^{\infty} F_n z^n.$$

Letting $F_0 = F_1 = 1$ and $F_{n+2} = F_{n=1} + F_n$, we represent the following as an OGF of a fibonacci sequence:

$$\begin{aligned}(1 - z - z^2) \sum_{n=0}^{\infty} F_n z^n &= \sum_{n=0}^{\infty} F_n z^n - \sum_{n=0}^{\infty} F_n z^{n+1} - \sum_{n=0}^{\infty} F_n z^{n+2} \\ &= \sum_{n=0}^{\infty} F_n z^n - \sum_{n=0}^{\infty} F_{n-1} z^n - \sum_{n=0}^{\infty} F_{n-2} z^n \\ &= F_0 + (F_1 - F_0)z + \sum_{n=2}^{\infty} (F_n - F_{n-1} - F_{n-2}) z^n.\end{aligned}$$

Thus, stating that $F_1 = F_0$ and $F_n = F_{n-1} - F_{n-2} z^n$ and presenting the sequence as

$$(1 - z - z2 \sum_{n=0}^{\infty} F_n z^n = F_0 = 1.$$

Therefore, proving that

$$\sum_{n=0}^{\infty} F_n z^n = \frac{1}{1 - (z + z^2)},$$

3.2 Exponential Generating Functions

Definition 3.3. Exponential Generating Functions are more representative of permutations within combinatorics because they are classified by labelled classes. Since the order of combinatorial objects matters in labelled classes, permutations is an essential part of enumeration of labelled classes through generating functions. Exponential Generating Functions provide a formal power series by encoding infinite sequences. They transform linear recurrence sequences of OGFs into using them for differential equations through formal power series.

Example 10. The Exponential Generating Function of a sequence $1, 1, 1, \ldots$ becomes

$$\sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x.$$

Supposing g_n to denote the set of size n, we let $g_0 = 1$. By the established cardinality within a combinatorial class, we have defined that $g_n = n$ which presents the EGF:

$$G(x) = \sum_{n=0}^{\infty} \frac{n!}{n!} x^n = \sum_{n=0}^{\infty} x^n = \frac{1}{1-x}.$$

Example 11. Let us suppose a fibonacci linear recurrence sequence that states $F_{n+2} = F_{n+1} + F_n$ and $F_0 = 0, F_1 = 1$. Defining

$$f(z) = \sum_{n \geq 1} F_n \frac{z^n}{n!}.$$

Multiplying our recurrence with by $\frac{z^n}{n!}$, it can be represented as

$$F(x) = \sum_{n \geq 0} F_{n+2} \frac{z^n}{n!} = \sum_{n \geq 0} F_{n+1} \frac{z^n}{n!} + \sum_{n \geq 0} F_n \frac{z^n}{n!},$$

which is then presented as a differential equation:

$$\frac{d^2}{dz^2} f(z) = \frac{d}{dz} f(z) + f(z).$$

Therefore, you understand that $f(0) = 0, f^1(0) = 1$.

4 Permutation Statistics

Definition 4.1. Permutations provide a mathematical technique to enumerate the ordering possibilities of a set of values. The use of permutations can be used to select a set of finite data from large datasets seen in algorithms analysed by analytic combinatorics. The use of permutations provides a generating function used to sort out small chunks of datasets within an infinitely increasing sequence. The use of these generating functions produced by permutations provide a statistical analysis for n as it reaches infinity.

Example 12. *Cycles* are a special type of permutation. Supposing π as a permutation of set $(1, 2, 3, \ldots, n)$, we consider the sequence

$$p_1, p_2 = \pi(p_1), \ldots, p_n = \pi(p_{n-1}).$$

When π is a cycle, it is only under the condition that $(p_1, p_2, \ldots, p_n)$ is equivalent to an ordered variation of $(1, 2, 3, \ldots, n)$. Using the rule of probabilities stated in 2 ($n-1$ choices for $p_2 = \pi(p_1)$, $n-2$ for $p_3 = \pi(p_2$, so on and so forth). The generating function for the number of cycles will be represented as

$$P(x) = \sum_{n-1}^{\infty} \frac{(x-n)!}{n!} x^n = \sum_{n=1}^{\infty} \frac{x^n}{n} = \log \frac{1}{1-x}.$$

In this paper, we explore permutations using generating functions under their use in limit shapes.

4.1 Erdos Szekeres Theorem

Theorem 4.2. *$\sigma \epsilon S_n$ and $n > rs$ for some integers $r, s \epsilon \mathbb{N}$, then $D(\sigma) > s$ or $L(\sigma) > r$.*

Proof. Using a variation on the permutation statistics $D(\cdot)$ and $L(\cdot)$, we state that for each $1 \leq k \leq n$, we let each $L_k(\sigma)$ and $D_k(\sigma)$ denote the maximal length of an increasing and decreasing subsequence respectivvely of σ that would end with $\sigma(k)$. Considering the n pairs in this dataset restricted by conditions, we understand that all pairs are distinct.

Since we can append an increasing subsequence σ that ends with $\sigma(j)$ to $\sigma(k)$ and has length $L_j(\sigma)$ to $\sigma(k)$, we understand that for every $1 \leq j < k \leq n$, $L_j(\sigma) < L_k(\sigma)$ under the condition that $\sigma(j) < \sigma(k)$. Alternatively, if $\sigma(j) > \sigma(k)$, then the increasing subsequence can be made longer by appending $\sigma(j)$. Therefore, we can find that $D_j(\sigma) < D_k(\sigma)$.

The conclusion we derive from this is that, contrary to the idea that $n > rs$, for some $1 \leq k \leq n$, either $L_k(\sigma) > r$ or $D_k(\sigma) > s$. Looking at this theorem's construction within random square young tableaux provides a permutation set with an interesting structure applicable to limit shapes. The strength of the theory that $n > rs$ is also highlighted in this construction.

□

Let us now look at the lower bound of l_n by denoting $l_n = \mathbb{E}L(\sigma_n)$.

Lemma 4.3. *For all $n \geq 1$, we use*

$$l_n \geq \sqrt{(}n)$$

Proof. For every permutation, we can say that for $\sigma \epsilon S_n$ we have $L(\sigma)D(\sigma) \geq n$. Due to symmetry, if l_n is the average for all values across $L_(\sigma)$ in $\sigma \epsilon S_n$, it will also become the average for all values across $D_(\sigma)$. Then, it is also equal to:

$$l_n = \frac{1}{n!} \sum_{\sigma \epsilon S_n} \frac{L(\sigma) + D(\sigma)}{2} = \mathbb{E}(\frac{L(\sigma_n) + D(\sigma_n)}{2}).$$

Using the inequality of arithmetic and geometric methods, we get

$$l_n \geq \mathbb{E}(\sqrt{L(\sigma_n)D(\sigma_n)}.$$

□

This provides us with the correct magnitude for l_n, however with the wrong constant. Now we look at the upper bound case of l_n under the same conditions.

Lemma 4.4. *As $n \to \infty$,*

we get

$$\limsup_{n\to\infty} \frac{l_n}{\sqrt{n}} \le e.$$

Proof. For every value $1 \ge k \ge n$, we denote all increasing subsequences of length k for random permutations σ_n as $X_{n,k}$. We will now calculate the expected value of $X_{n,k}$ keeping in mind that this value is equivalent to the sum of all values across $\binom{n}{k}$ subsequences of length k with an increasing subsequence presented by the probability $1/k!$.

This would give us

$$\mathbb{E}(X_{n,k} = \frac{1}{k!}\binom{n}{k}.$$

This would bound the possibility of k being the minimum length for $L(\sigma_n)$ and we denote it with

$$\mathbb{P}(L(\sigma_n) \ge k) = \mathbb{P}(X_{n,k} \ge 1) \le \mathbb{E}(X_{n,k}\frac{1}{k!}\binom{n}{k}$$

$$= \frac{n(n-1)\dots(n-k+1)}{(k!)^2} \le \frac{n^k}{(k)^{2k}}.$$

We can then fix $\delta > 0$ whilst taking $k = \lceil (1+\delta)e\sqrt{n} \rfloor$ so that we get

$$\mathbb{P}(L(\sigma_n) \ge k) \le \frac{n^k}{k/e^{2k}} \le (\frac{1}{1+\sigma})^{2k} \le \frac{1}{1+\sigma}^{2(1+\sigma)e\sqrt{n}},$$

we see a bound converging at 0 at an exponential rate for $\sqrt{n}$ when $n \to \infty$ which can be followed as the following equation where a positive constant c depends on σ:

$$l_n = \mathbb{E}(L(\sigma_n)) \le \mathbb{P}(L(\sigma_n) < k)(1+\sigma)e\sqrt{n} + \mathbb{P}(L(\sigma_n) \ge k)n \le (1+\sigma)e\sqrt{n} + O(e^{-c\sqrt{n}}).$$

Therefore, it is proven that σ is an arbitrary number.

□

This proves that these values not only apply as an average value but rather a typical value of $L(\sigma_n)$. These bounds also show that changes away from the typical values behave similar to exponential function of $\sqrt{n}$ in terms of decay.

4.2 Robinson Schnested Algorithm

The combinatorial power of algorithms such as the patience sorting algorithm are represented magnificently in the Robinson Schnested Algorithm. To understand it, we must note that the array of stacks produced by algorithms such as the patience sorting algorithm requires much more data than required to compute $L(\sigma)$*, which would be the amount of stacks.*

In these stacks, the topmost numbers are of utmost importance in the execution such that the first row in the array determine the placement of a new number. The other rows are simply used as input for further sorting and the process continues per row. We must then consider the processing on the addition of a new number and now it is determined to either exist next to existing arrays or start a new one. The latter is a simple process since the number creates a new array and the algorithm moved into sorting the next number in the dataset. If the number is to place itself at the top of an existing array then the bottom numbers are pushed down and designated the task to find and replace their array placement (which is usually right below their previous position for each number added). If we were to consider the event in which the number places itself in the second row then the numbers below it recursively need to reposition themselves accordingly and the process stops once a value places itself in a new unoccupied space. This creates a cycle of permutations.

The following figure shows an array sequence in which number placements shift to accommodate new values within the permutation:

$$4 \rightarrow \begin{array}{l}1\\4\end{array} \rightarrow \begin{array}{ll}1&2\\4&\end{array} \rightarrow \begin{array}{lll}1&2&7\\4&&\end{array} \rightarrow \begin{array}{lll}1&2&6\\4&7&\end{array}$$

$$\rightarrow \begin{array}{lll}1&2&5\\4&6&\\7&&\end{array} \rightarrow \begin{array}{llll}1&2&5&8\\4&6&&\\7&&&\end{array}$$

$$\rightarrow \begin{array}{lllll}1&2&5&8&9\\4&6&&&\\7&&&&\end{array} \rightarrow \begin{array}{lllll}1&2&3&8&9\\4&5&&&\\6&&&&\\7&&&&\end{array}$$

The computational technique of the Robinson Schensted algorithm showcases the refinement of array sorting through inputting permutations to produce statistical sorting and therefore, making the sorting algorithm much easier and swifter.

4.3 Square Young Tableaux

Suppose $n\epsilon N$*. A loose term for noting* n *as a sum of positive integers without the importance of ordering would be called a* ***partition****. We claim*

$$\lambda = \lambda_1, \lambda_2, \ldots, \lambda_k$$

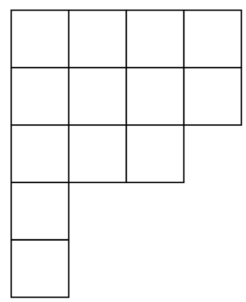
where all numbers are positive integers and these integers are then called parts of the partition.

This is an important detail because of its relevance to parts of a partition being determinants of row lengths in two-dimensional number arrays produced by algorithms. We denote λ as a partition of n with $\lambda \vdash n$. We also denote partition sets of n by $P(n)$.

For the demoted partition of n ($\lambda \vdash n$) we define the Young Tableaux

Definition 4.5. A young tableaux is a combinatorial object in which numbers become non decreasing along their lines and increasing along their columns. They are used to represent ordered datasets and therefore, represent permutations as ordered array.

An example is this figure of the Young Diagram representing the partition $(4, 4, 3, 1, 1)$



Young diagrams represent integer partitions and therefore are often identified through the partition they represent. If $\lambda \vdash n$ is denoted λ' as a conjugate partition of λ, its Young diagram is identified by following its column lengths, also stated as its **principal diagonal**. The conjugate partition for our Young tableaux figure representing $\lambda = (4, 4, 3, 1, 1)$ would become $\lambda' = (5, 3, 3, 2)$.

A Young tableaux is characteristic of filling cells in an ordered manner where all rows and columns are set in an increasing order which is also characteristic of the sorting algorithm we have looked at. We can then consider P as a young tableau, specifically the insertion tableau. We can also record a young tableau Q where P, if occupied by k during the kth insertion of numbers in the algorithm. Q would then be responsible for tracking the addition of cells as a shape "grows" to a numerical diagram from an empty one.

Theorem 4.6. *The Robinson Schensted algorithm is defined as a mapping of bijection between S_n and triple sets (λ, P, Q) by taking a permutation $\sigma \epsilon S_n$. This creates the identity:*

$$L(\sigma) = \lambda_1(lengthofthefirstrowof\lambda.$$

Further enumerating Young Tableaux of numerous shapes as they "grow" from empty diagrams, we can see that for each $n \geq 1$ we have

$$\sum_{\lambda \vdash n} d_\lambda^2 = n!$$

in that this sum applies across every partition of n.

Using the Erdos Szekeres theorem with Young Tableaux, we focus on permutation statistics aiding limit shapes.

For integers $m, n \geq 1$ *we can state the permutation as* $\sigma \epsilon S_n$ *in which* $N = mn, L(\sigma) = n and D(\sigma) = m$ *where* $D(\sigma)$ *would represent the maximal length of decreasing subsequences as was stated previously. Supposing the case when* $m = n$, *we state the permutation* $\sigma \epsilon ES_{n,n}$ *and denote it by* ES_n.

Let $\sigma \epsilon ES_{m,n} and \lambda, P, Q$ *be our Young diagram for this case under the Robinson Schensted algorithm.*

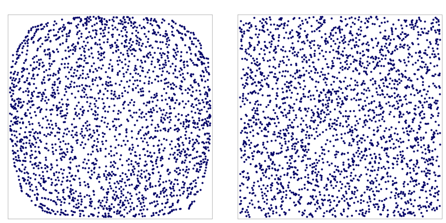
Theorem 4.7. *The Robinson Schensted algorithm bijectively uses the pair of Young Tableaux* (P, Q) *to map the set* $ES_{m,n}$

$$|ES_{m,n}| = d_{m,n}^2 = \frac{(mn)!}{\Pi_{i=1}^{m} \Pi_{j=1}^{n} (i + j - 1)}^{2}.$$

The Erdos-Szekeres permutations are bijectively mapping the set of the $n \times n$ *in the Young tableaux which is shown by*

$$|ES_n| = d_{n,n}^2 = \left(\frac{(n^2)!}{1 \cdot 2^2 \cdot 3^3 n^n (n+1)^{n-1} (n+2)^{n-2} (2n-1)^1}\right)^2.$$

This theorem provides a structure for Erdos Szekeres permutations generating from Young tableaux. Due to the connection to square Young tableaux, we can consider a random Erdos Szekeres permutation by randomly selecting two square Young Tableaux. Using a uniformly random permutation $S_2 500$, *we use a figure to represent the density and graphic plotting of a random permutation produced under these conditions.*



(a) (b)

It also showcases the asymptotic progression of permutations as $n \to \infty$.

4.4 Limit Shape theorem and random permutations

Theorem 4.8. *We now use the limit shape theorem to produce random Erdos Szekeres permutations and utilise analytic combinatorics within limit shapes through permutation statistics with the use of square Young Tableaux. Suppose* σ_n *denotes a permutation that is randomly chosen in a uniform manner from*

ES_n for each n. Supposing the denotation of A_n as $(j, \sigma_n(j)) : 1 \le j \le n^2$. We can represent the set $A_\infty \subset \mathbb{R}^2$ with the following

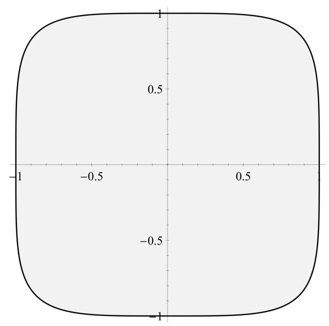
$$A_\infty = (x, y)\epsilon\mathbb{R}^2 : (x^2 - y^2)^2 + 2(x^2 + y^2) \le 3$$

$$= (x, y)\epsilon(R)^2 : |x| \le 1, |y| \le \sqrt{x^2 - 1 + 2\sqrt{1 - x^2}}.$$

As $n \to \infty$, a convergence of probability for A_∞ occurs under two true conditions:

(a) For any $\epsilon > 0, \mathbb{P}(\tilde{A}_n \subset (1+\epsilon)A_\infty) \to 1$ as $n \to \infty$
(b) For any open set $U \subset A_\infty, \mathbb{P}(\tilde{A}_n \cap U \neq \emptyset \to 1 \to \infty$.

The following figure represents the limit shape formed by A_∞.



In order to understand limit shapes from random permutations, we need to truly understand the connection between the Young tableau and permutations. In this case we will use the Erdos Szekeres permutations and the square Young tableaux. Looking into the tableau, we need to look at the different visualisations of the tableau. The first interpretation is the usual array format of the tableau of numbers $(t_{i,j})_{i,j=1}^n$, and creates a graph that resembles a "stepped surface" due to the height variations of unit cubes covering the square tableau. The other representation, alternatively showcases the "growth" of the tableau by tracking the diagram as it grows from an empty set to its full form and records it as different diagrams. This interpretation visualises the tableau as an increasing sequence of Young diagrams

$$\Phi = \lambda^{(0)} \nearrow \lambda^{(1)} \nearrow \ldots \nearrow \lambda^{(n)} = \lambda.$$

The diagram produced from this would show a path of all cells of λ where the tableau entries are $\le k$. Through the latter interpretation, we visualise the tableau through the traced path of the tableau presented by the data which in turn shows the growth of the tableau through the records of the tableau's growth from an empty diagram to a fully formed one.

Theorem 4.9. *For all values of $n \geq 1$, we suppose that $T_n = (t^n_{i,j})^n_{i,j=1}$ is a consistently random Young tableau of a shape denoted by (n, n). For all values of $\epsilon > 0$, we can write*

$$\mathbb{P}[\max_{1\leq i,j\leq n} |n^{-2}t^n_{i,j} - S(i/n, j/n)| > \epsilon] \to 0 as n \to \infty.$$

Using 4.8, we prove this theorem.

Proof. Using the bijection claim, we use two Young tableau identified by their uniformly random permutations σ_n which are $P_n = (p^n_{i,j})^n_{i,j=1}, Q_n = (q^n_{i,j})^n_{i,j=1}$. The set can then also be written as

$$A_n = (q^n_{i,j}, p^n_{n+1-i,j}) : 1 \leq i, j \leq n.$$

Using this theorem, we look at all points $n^{-2}(q^n_{i,j}, p^n_{n+1-i,j})$ that become uniformly close to point $(S(x, y), S(1 - x, y))$ where $x = i/n$ and $y = j/n$ as $n \to \infty$. We, therefore, define the set as

$$A'_\infty = (2S(x, y) - 1, 2S(1 - x, y) - 1) : 0 \leq x, y \leq 1.$$

This denotation of set A_n as $n \to \infty$, the scaled set values with intersect with a subset of A'_∞.

The set A'_∞ becomes a mapped square of $[0, 1] \times [0, 1]$

$$\Phi : (x, y) \to 2S(x, y) - 1, 2S(1 - x, y) - 1$$

Φ would map the four boundaries of the squares defined by:

$$(-\sqrt{1 - t^2}, -\sqrt{2t - t^2})_{0\leq t\leq 1},$$

for every boundary. Such that these curves with set parameters for the boundaries of the set A_∞

We can see now that $(x, y) \to 2S(x, y) - 1$ is subsequently increasing in x and y both and that $(x, y) \to 2S(1 - x, y)$ is subsequently increasing in y and decreasing in x and therefore, Φ becomes one-to-one. As a result, the square of the mapping produced by $[0, 1]^2$ is done under Φ. Thus, proving $A'_\infty = A_\infty$. □

The use of the permutation ES_n in two uniformly random Young tableaux showcases the use of random permutations to compute, sort and arrange datasets in the parameters of limit shape construction.

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