The Birkhoff Ergodic Theorem

Ezra Furtado-Tiwari

July 10, 2022

Ergodic Theory

Ergodic theory studies the impacts of transformations on dynamic systems. Using tools from measure theory, we can prove many powerful results for almost all numbers in certain sets. For instance, here are some questions we can answer using ergodic theory:

1. Given a random number in $[0, 1)$, what fraction of its decimal digits are 1?
2. What is the probability that a power of 2 starts with 7?
3. What is the probability that a random coefficient chosen in the continued fraction of a number in $[0, 1)$ is k?

The Lebesgue Outer Measure

Definition

We define the *Lebesgue outer measure* of a set $A \in \mathbb{R}$ to be

$$\lambda(A) = \inf \left\{ \sum_{j=1}^{\infty} |I_j| : A \subset \bigcup_{j=1}^{\infty} I_j \right\}$$

where the I_j are a sequence of bounded intervals.

Measures

Definition

A *measure* on a σ-algebra $\mathcal{S}$ is a function $\mu : \mathcal{S} \to [0, \infty)$ such that $\mu(\emptyset) = 0$ and

$$\mu\left(\bigsqcup_{j=1}^{\infty} A_j\right) = \sum_{j=1}^{\infty} \mu\left(A_j\right).$$

Measure Spaces

Definition

A *measure space* is a triple $(X, \mathcal{S}, \mu)$ where X is a nonempty set, $\mathcal{S}$ is a σ-algebra on X, and μ is a measure on $\mathcal{S}$. A *probability space* satisfies $\mu(X) = 1$.

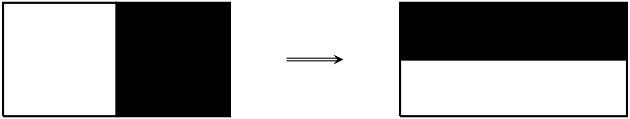
The Baker's Transformation

Definition

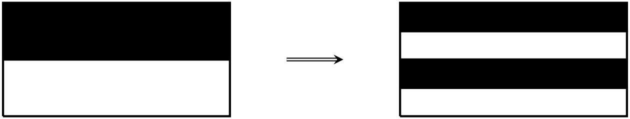
The *Baker's transformation* is the transformation

$$T(x,y) = \begin{cases} \left(2x, \frac{y}{2}\right) & \text{if } 0 \leq x < \frac{1}{2} \\ \left(2x-1, \frac{y+1}{2}\right) & \text{if } \frac{1}{2} \leq x < 1. \end{cases}$$

The Baker's Transformation



The Baker's Transformation



Invariance

Definition

Let $(X, \mathcal{S}, \mu)$ be a measure space, and let $T : X \to X$ be a transformation. A subset $A \subset X$ is *strictly invariant* if

$$T^{-1}(A) = A.$$

Ergodicity

Definition

Let $(X, \mathcal{S}, \mu)$ be a measure space, and let T be a measure-preserving transformation. T is said to be *ergodic* if when A is a strictly invariant measurable set, either $\mu(A) = 0$ or $\mu(X \setminus A) = 0$.

The Lebesgue Integral

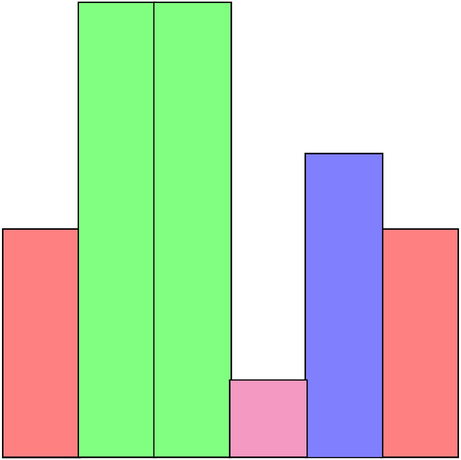
Definition

Let f be a nonnegative function. Then

$$\int f \, d\mu = \sup \left\{ \int g \, d\mu : g \text{ is simple and } 0 \leq g \leq f \right\}$$

is called the *Lebesgue integral* of f.

The Lebesgue Integral



The Birkhoff Ergodic Theorem

Theorem (Birkhoff Ergodic Theorem)

Let $(X, \mathcal{S}, \mu)$ be a measure space and let T be a measure-preserving transformation. Then if $f : X \to \mathbb{R}$ is a Lebesgue integrable function,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} f\left(T^i(x)\right) = \int f \, d\mu.$$

Normal Numbers

Definition

Let x be a real number such that $x \in [0, 1)$. We let

$$M(x, b) = x \pmod{1}_b = b \cdot x - \lfloor b \cdot x \rfloor .$$

Definition

We let a real number $x \in [0, 1)$ be *normal* in base b if for every $y \in \{0, 1, \ldots, b-1\}$, we have

$$\frac{1}{n} \lim_{n \to \infty} \sum_{i=0}^{n-1} M^i(x, b) \mathbb{1}_{[\frac{y}{b}, \frac{y+1}{b})} = \frac{1}{b}.$$

Normal Numbers

Theorem

Almost every number $x \in [0, 1)$ *is normal in every base* b.

Normal Numbers

Consider the number

$$0.012345678910111213\ldots.$$

For the sake of approximation, we iterate this number until 99. This number is known to be a normal number.

Normal Numbers

Consider the binary representation of the number in the previous slide. Truncating this number at different lengths and checking the number of 1s gives us the following results:

100	46
150	67
200	94.

Interestingly enough, this number appears to be close to $\frac{1}{2}$ of the length of the number.

Normal Numbers

A similar process for base 3, instead considering the number of 1s and 2s gives the following result:

100	23	36
150	43	48
200	60	65.

Again, we end up with $\sim \frac{1}{3}$ of each of the digits, suggesting that this number is normal in base 3 as well.

Continued Fractions

Definition

We define the *Gauß map* to be the transformation

$$T(x) = \begin{cases} \frac{1}{x} - \left\lfloor \frac{1}{x} \right\rfloor & x \neq 0 \\ 0 & x = 0. \end{cases}$$

Continued Fractions

Proposition

Let $x \in [0,1)$ *be a real number and let*

$$[a_0, a_1, a_2, \ldots, a_n, \ldots]$$

be the coefficients of its continued fraction representation. Then

$$a_{n+1} = \left\lfloor \frac{1}{T^n(x)} \right\rfloor,$$

where $T(x)$ *is the Gauß map.*

Continued Fractions

Theorem

Let x be a real number such that $x \in [0, 1)$, and let $[a_0, a_1, a_2, \ldots, a_n, \ldots]$ be its continued fraction representation. Then

$$\mathbb{P}(a_i = k) = \frac{1}{\log 2} \log\left(\frac{(k+1)^2}{k(k+2)}\right).$$

Thus we have $\approx 41.5037\%$ 1s, $\approx 16.9925\%$ 2s, $\approx 9.31094\%$ 3s, $\approx 5.88937\%$ 4s, $\approx 4.0642\%$ 5s, and so on.

Conclusion

Thank You!