Riemann Surfaces, Teichmüller Spaces, and Fenchel-Nielson Coordinates

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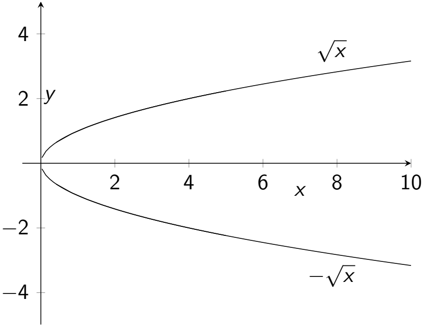
The Square Root Function

Example

Graph $y = \sqrt{x}$

Equivalent to solving for y in $x = y^2$.



The Square Root Function

For z with Im(z) $\in \mathbb{R}$ and Re(z) $\in \mathbb{R}$:

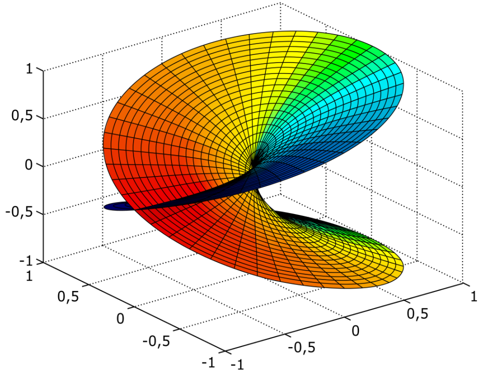
$$w = z^{\frac{1}{2}}$$

$$w = \begin{cases} \sqrt{z} = \sqrt{r}e^{\frac{i\theta}{2}} \\ -\sqrt{z} = -\sqrt{r}e^{\frac{i\theta}{2}} \end{cases}$$

How do we graph a complex input and a complex output?

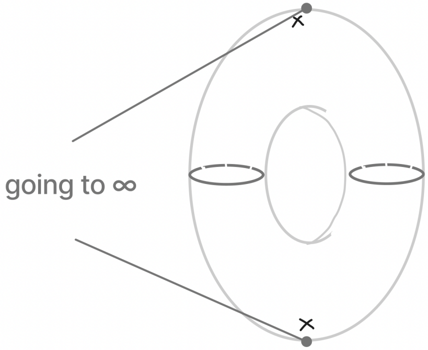
The Square Root Function

Let $(x, y, z) = (\mathrm{Re}(z), \mathrm{Im}(z), \mathrm{Re}(w))$.



An Interesting Example

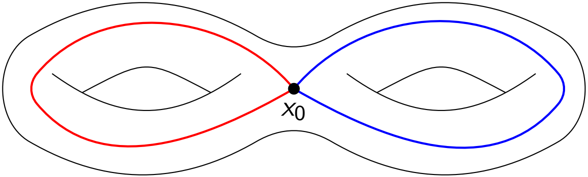
$w = \sqrt{(z^2 - 1)(z^2 - k^2)}$ where $k \in \mathbb{C}$, $k \neq \pm 1$ and $(z, w) \in \mathbb{C}^2$



Marked Riemann Surfaces

Definition

The **fundamental group** is the group of equivalence classes under homotopy of the loops connected in the space. We can denote the fundamental group as $\pi_1(X, x_0)$ for a topological space X with base point x_0.



Marked Riemann Surfaces

For a Riemann surface with genus g, the fundamental group consists of $2g$ elements. These elements make up a **marking** on a Riemann surface.

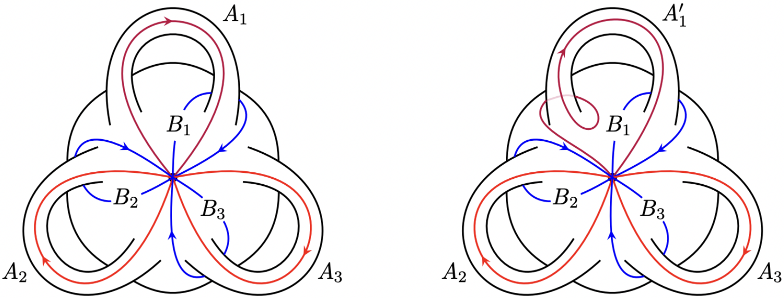
Definition

For a Riemann surface R of genus g, a **marking** on R is defined as $\Sigma_g = \{|A_j|, |B_j| \,|\, j = 1, 2, ..., g\}$ of $\pi_1(R, p)$. We denote a **marked Riemann surface** as (R, Σ_g).

Marked Riemann Surfaces

Example

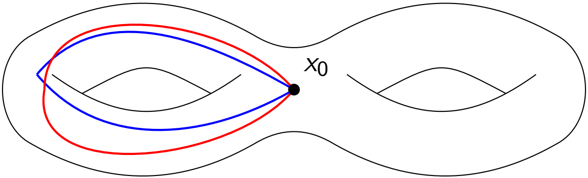
The collection of paths associated with markings
$\Sigma_g = \{|A_1|, |A_2|, |A_3|, |B_1|, |B_2|, |B_3|\}$ (left) and
$\Sigma_g = \{|A_1'|, |A_2|, |A_3|, |B_1|, |B_2|, |B_3|\}$ (right) on surfaces of genus 3.



Marked Riemann Surfaces

Two markings are considered equivalent if they are homeomorphic to each other.

Example



Marked Riemann Surfaces

Definition

Two marked Riemann Surfaces (R, Σ_{g_1}) and (S, Σ_{g_2}) are considered **marking equivalent** if there exists an isometry $m : R \to S$ such that Σ_{g_2} and $m \circ \Sigma_{g_1}$ are isotopic.

Teichmüller Space

Definition

We define the **Teichmüller space** of a Riemann surface S of genus $g \geq 2$ as

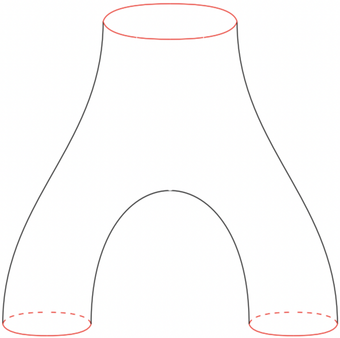
$$T_g = \{(S, \Sigma_g)\}/\sim$$

where Σ_g is the marking on S.

Fenchel-Nielson Coordinates

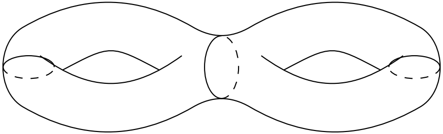
Definition

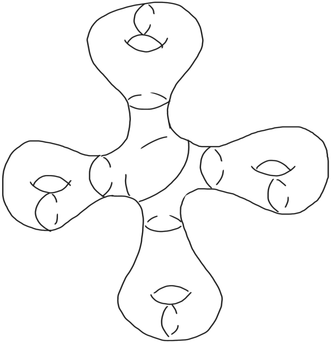
A **pair of topological pants** is a genus zero surface with three boundaries.



Fenchel-Nielson Coordinates

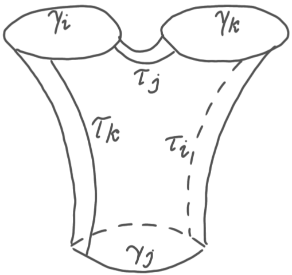
Example



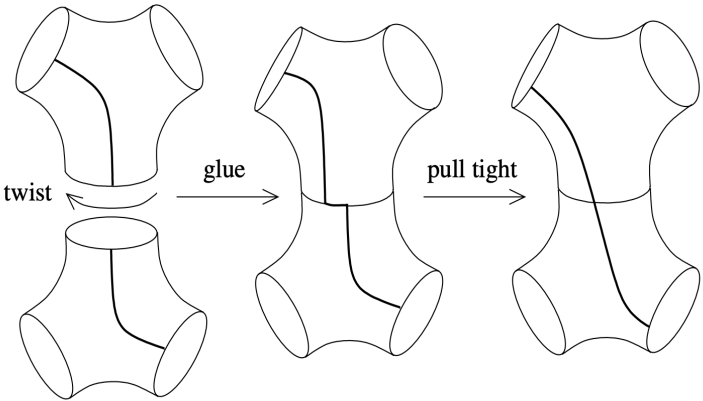


Fenchel-Nielson Coordinates

A **geodesic** is a length minimizing curve. Assume all of the boundaries $\{\gamma_1, \gamma_2, ..., \gamma_{3g-3}\}$ are geodesics. Define the lengths $\{\tau_1, \tau_2, ...\tau_{3g-3}\}$ as the shortest paths between these boundaries.



Fenchel-Nielson Coordinates



Fenchel-Nielson Coordinates

Definition

For a surface S of $g > 1$, S can be decomposed in $2g - 2$ pairs of pants $\{P_1, P_2, ...P_{2g-2}\}$ by closed geodesics $\{\gamma_1, \gamma_2, ..., \gamma_{3g-3}\}$. The **Fenchel-Nielson coordinates** of S in a Teichmüller space $T_{g,n}$ are given by:

$$\{(|\gamma_1|, \theta_1), (|\gamma_2|, \theta_2), ..., (|\gamma_{3g-3}|, \theta_{3g-3})\}.$$

T_S is Homeomorphic

Theorem

Let $g \geq 2$ for a surface S. The map

$$FN : T_S \to \mathbb{R}_+^{3g-3} \times \mathbb{R}^{3g-3}$$

defined by

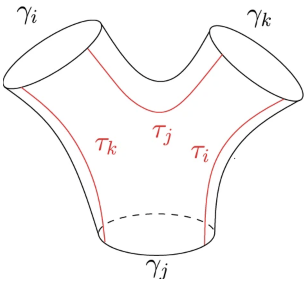
$$FN(X) = (|\gamma_1|, |\gamma_2|, ...|\gamma_{3g-3}|, \theta_1, \theta_2, ..., \theta_{3g-3})$$

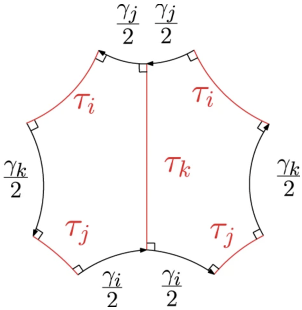
is a homeomorphism.

Proof

Definition

Given a pair of pants with geodesic boundaries there exists three geodesic arcs that join the cuffs perpendicular to their endpoints. These arcs collectively are called the **seams** of the pants.





Proof

Proposition

For $l_1, l_2, l_3 \in \mathbb{R}_+$, there exists a unique hyperbolic right-angled hexagon with alternating edge lengths (l_1, l_2, l_3).

Proposition

Given a surface S with genus g and pants decomposition P, the surface can be decomposed in $2g - 2$ pairs of pants.

Proposition

There are $3g - 3$ curves in a pants decomposition for a surface S with genus g.

Proof

Any point on the Teichmüller space of a Riemann surface S of genus g can be specified by $3g - 3$ nonnegative real numbers (lengths of hyperbolic seams) and $3g - 3$ real numbers (twisting angles). Therefore:

$$T_g \to \mathbb{R}_+^{3g-3} \times \mathbb{R}^{3g-3}$$

and the Fenchel-Nielson coordinates specify this point. This concludes our proof.

Purpose

1. T_S is an easier, more understandable space than the moduli space
2. The study of T_S - Teichmüller theory - a whole field itself
   1. The use of metrics: Weil-Petersson metric
   2. Study of the shape and distance

Thank you

Thank you for listening!