Cayley graphs and Cayley representations

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July 6, 2022

Outline

1. Background: what is a group, Cayley graph, group action
2. Simple results: regularity, vertex-transitivity
3. Advanced results, Cayley representations
4. Further reading

Definition: Group

A group is a set of elements G with a binary operation $\cdot$ satisfying the following properties:

- Closed: $a \cdot b \in G$ for all $a, b \in G$.
- Associativity: $a \cdot (b \cdot c) = (a \cdot b) \cdot c$ for all $a, b, c \in G$.
- Identity $e \in G$ such that $a \cdot e = e \cdot a = a$ for all $a \in G$.
- Inverse element a^{-1} for $a \in G$ such that $a \cdot a^{-1} = a^{-1} \cdot a = e$.

Example 1

The set of integers mod 6, $\mathbb{Z}_6$, is a group.

Example 2

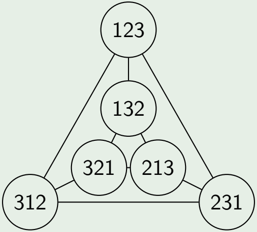
The empty set is not a group.

Definition: Cayley graph

Let G be a group, and let $S \subseteq G$ be a generating set of G. The Cayley graph $\Gamma(G, S)$ is a simple connected graph with vertex set G and edges $\{g, gs\}$ for all $g \in G$.

Example 3

Symmetric group S_3, $S = \{132, 312\} \subset S_3$ generates S_3. The Cayley graph $\Gamma(S_3, S)$ is:



Group action

Definition.

A group action by a group G on a set X is a homomorphism from G to the set of permutations of X.

Example 4

The symmetric group of degree 3, S_3, naturally acts on the ordered triple $(1, 2, 3)$ by permuting its elements. S_3 even acts on the ordered quadruple $(1, 2, 3, 4)$ by permuting the first 3 elements.

Example 5

The group of Rubik's cube moves acts in the usual way on the set of corners of the cube.

Regularity

A graph is said to be regular if each vertex has the same number of neighbors(degree). We can show that all Cayley graphs are regular:

Proof.

Let G be a group and S a generating subset of G. Let s be an element of S. Then, for every $s \in S$ in the Cayley graph $\Gamma(G, S)$,

- $s^2 \neq e$: Exactly two edges generated by s connecting to any given $g \in G$: $\{g, gs\}$ and $\{gs^{-1}, g\}$. Proof by contradiction: suppose $\{g, h\}$ is a distinct edge generated by s for some element h. $hs = g$ or $gs = h$, so $h = gs$ or $h = gs^{-1}$.
- $s^2 = e$: Exactly one edge generated by s connecting to any given $g \in G$.

Therefore, the degree of every vertex in a Cayley graph is the same, so $\Gamma(G, S)$ is regular. ■

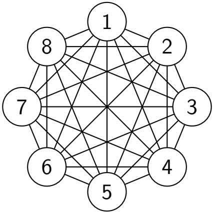
Graph Automorphism

Definition.

A graph automorphism of a graph $X = (V, E)$ is a permutation $f : V \to V$ which preserves edges and nonedges:

$$\{u, v\} \in \mathrm{E}(X) \iff \{f(u), f(v)\} \in \mathrm{E}(X)$$

Example: Every permutation of the vertices of the complete graph K_8 is an automorphism.



Vertex Transitivity

Definition.

A graph is said to be vertex-transitive if there exists an automorphism mapping u to v for any two vertices u, v.

We can prove that all Cayley graphs are vertex-transitive:

Proof.

The permutation $\ell_g : G \to G (g \in G)$ which maps $a \mapsto ga$ for all $a \in G$ is an automorphism:

$$\{ga, gas\} \in \mathrm{E}(\Gamma(G, S)) \iff \{a, as\} \in \mathrm{E}(\Gamma(G, S))$$

Then let $u, v \in G$. Then, $\ell_{vu^{-1}}(u) = vu^{-1}u = v$, so there exists an automorphism from any vertex to any other vertex. ∎

Sabidussi on automorphism groups

Let $\Gamma(G, S)$ be a Cayley graph. Let ℓ be the group action by G on $\Gamma(G, S)$ mapping $g \mapsto \ell_g$ as defined in the previous slide. Then,

Theorem 6 (Sabidussi '64 [1])

A connected graph $\Gamma = (G, E)$ is a Cayley graph of the group G if and only if $\ell G \leq \mathrm{Aut}(\Gamma)$.

This theorem is extremely useful, since it provides an alternative condition for testing whether a given graph is Cayley or not.

Cayley representation problem

The pair (G, S) is called a Cayley representation of a graph Γ if $\Gamma \cong \Gamma(G, S)$.

Cayley representation problem

Given a Cayley graph Γ, determine all Cayley representations (G, S).

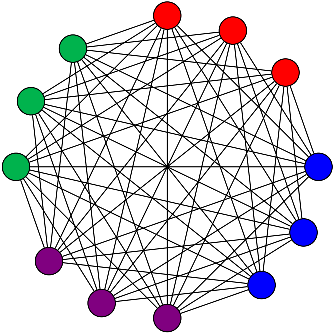
The problem is an active area of study. Current results include

Theorem 7 (Li '02 [2])

A group G has a Cayley graph isomorphic to $K_{m;d}$(the complete d-partite graph with parts of order m) if and only if G has order md and has a subgroup G_0 of order m, and the generating set is $G \setminus G_0$.

Cayley representation problem

The following graph can be represented as a Cayley graph of $G = \mathbb{Z}_{12}$ and $S = G \setminus \{0, 4, 8\}$.



The parts represent $\{0, 4, 8\}$, $\{1, 5, 9\}$, $\{2, 6, 10\}$, and $\{3, 7, 11\}$.

Further reading

Keith Conrad's survey on group actions:
https://kconrad.math.uconn.edu/blurbs/grouptheory/gpaction.pdf

Cai-Heng Li's survey on Cayley graphs and representation:
https://www.sciencedirect.com/science/article/pii/S0012365X01004381

References I

[1] Gert Sabidussi. Vertex-transitive graphs. *Monatshefte für Mathematik*, 68(5):426–438, 1964.

[2] Cai Heng Li. On isomorphisms of finite cayley graphs—a survey. *Discrete mathematics*, 256(1-2):301–334, 2002.