Cayley Graphs – Euler Circle Report

Ethan Liu

June 2022

Abstract

1 Introduction

Cayley graphs, named in honor of mathematician Arthur Cayley, are an important concept linking group theory and graph theory. Mathematicians have extensively investigated the isomorphisms of Cayley graphs, seeking to apply group theory results to specific classes of graphs, and finding Cayley graph representations continues to be an active area of study in graph theory.

In this paper, we start by summarizing the basics of group theory and graph theory, as well as group actions, orbits, stabilizers, and group and graph automorphisms. We then move on to Cayley graphs and their general properties, after which we describe specific vertex-transitive graphs that can be represented as Cayley. We conclude by looking into the problem of Cayley representation, specifically the number and type of Cayley representations that a given graph can have.

2 Background

Analyzing Cayley graphs calls for basic knowledge of group theory and graph theory: only then can we fully visualize the concepts involved with Cayley graphs both through a group theory lens and from a graph theory perspective. In this section, we will provide relevant definitions and examples for groups, group actions, isomorphisms, automorphisms, and graphs.

2.1 Group theory basics

To understand Cayley graphs, we must first familiarize ourselves with some fundamental definitions from group theory.

Definition 2.1. A group is a set of integers G with a binary operation $\cdot$ satisfying the following properties:

- G is closed under $\cdot$: $a \cdot b \in G$ for all $a, b \in G$.
- $\cdot$ is associative: $a \cdot (b \cdot c) = (a \cdot b) \cdot c$ for all $a, b, c \in G$.
- There exists an identity element $e \in G$ such that $a \cdot e = e \cdot a = a$ for all $a \in G$.
- For all elements $a \in G$, there exists an inverse element a^{-1} such that $a \cdot a^{-1} = a^{-1} \cdot a = e$.

Notice that the empty set is not a group, since a group must have an identity element.

Example 2.1. Consider the set S_n of permutations of the set $\{1, 2, \ldots, n\}$. Notice that this set is a group under function composition:

- S_n is closed under $\circ$: $a \circ b \in S_n$ for all $a, b \in S_n$.
- Function composition is associative: $a \circ (b \circ c) = (a \circ b) \circ c$ for all $a, b, c \in S_n$.
- The identity element $e = 12 \ldots n$ satisfies $a \circ e = e \circ a = a$ for all $a \in S_n$.
- For all elements $a \in S_n$, there exists an inverse element a^{-1}, since permutations are bijective and therefore have inverses which are also permutations

We call this group the symmetric group: it contains all symmetries of a set of size n.

It is important to keep in mind that when the symbol $\leq$ is used in the context of comparing two groups, for example $G \leq H$ for two groups G and H, then it means that G is a subgroup of H.

2.2 Mappings of groups

We first define group homomorphisms and symmetries, as well as two special types of homomorphisms called isomorphisms and automorphisms. Then we define the symmetric group.

Definition 2.2. Let G and H be groups with binary operations $\cdot$ and $*$, respectively. Then, a group homomorphism is a function $f : G \to H$ such that for any two elements $g_1, g_2 \in G$,

$$f(g_1 \cdot g_2) = f(g_1) * f(g_2)$$

In other words, a group homomorphism is a function that respects binary operations. Consequently, f preserves the identity element:

$$f(e_G) = e_H$$

where e_G and e_H are the respective identities of G and H, as well as inverses:

$$f(g^{-1}) = f(g)^{-1}$$

where g is any element of G.

Definition 2.3. A group isomorphism is a bijective group homomorphism between the two groups. Due to the one-to-one nature of group isomorphisms, if there exists an isomorphism between two groups G and H then we say G and H are isomorphic: their internal structures are completely identical.

We use the notation $G \cong H$ to say that G is isomorphic to H.

Definition 2.4. A group automorphism is an isomorphism from a group G to itself. An automorphism is often expressed in the form of a permutation of the elements of G.

Example 2.2. Let $\mathrm{GL}_2(\mathbb{R})$ be the group of 2x2 invertible real-valued matrices, and let $(\mathbb{R}, \times)$ be the multiplicative group of nonzero real numbers. Then, there exists a homomorphism $f : \mathrm{GL}_2(\mathbb{R}) \to (\mathbb{R}, \times)$ that maps every matrix to its determinant. Notice that for any two matrices $A, B \in \mathrm{GL}_2(\mathbb{R})$,

$$\det(AB) = \det(A) \times \det(B)$$

Also notice that $\det(I) = 1$, so f preserves identity.

Example 2.3. For every group G there exists an automorphism mapping every element to itself: this is known as the trivial automorphism.

Example 2.4. Let $G = \mathbb{Z}^2$ be the group of ordered pairs of integers under componentwise addition. Then, there exists an automorphism $f : G \to G$ that maps (x, y) to (y, x) for all $(x, y) \in G$.

Definition 2.5. Let M be a set. Then, the group of all bijective mappings from M to itself is known as the symmetric group $\mathrm{Sym}(M)$. On the other hand, the group of all automorphisms of M is known as the automorphism group $\mathrm{Aut}(M)$.

Elements of groups have been shown to correspond with permutations of another space. For instance, Cayley's theorem states the following:

Theorem 2.1 (Cayley's Theorem)**.** Every finite group G can be embedded into a symmetric group.

Proof. For any element $g \in G$, let $\ell_g : G \to G$ be the mapping that maps $x \in G$ to gx. Notice that such a mapping is injective: if two elements $a, b \in G$ both mapped to the same element through ℓ_g, then that would mean $ga = gb = h$ for some $h \in G$, so $a = g^{-1}h = b$. Therefore, ℓ_g is a bijection from G to itself, so $\ell_g \in \mathrm{Sym}(G)$.

Now we show that the mapping $\ell : G \to \mathrm{Sym}(G)$ which maps every $g \in G$ to $\ell_g \in \mathrm{Sym}(G)$ is an isomorphism. Notice that for any two $g_1, g_2 \in G$, $\phi_{g_1} \circ \phi_{g_2} = \phi_{g_1 g_2}$. Therefore, $\ell(g_1 g_2) = \ell(g_1) \circ \ell_{g_2}$, so ℓ is a homomorphism. Now we must show that ℓ is injective: suppose there exist two elements $g_1, g_2 \in G$ such that $\ell(g_1) = \ell(g_2)$. Then, $\ell_{g_1} = \ell_{g_2}$, so $\ell_{g_1}(e) = \ell_{g_2}(e)$, so $g_1 e = g_2 e$, so $g_1 = g_2$.

Therefore, ℓ is an isomorphism, so G can be embedded as a subgroup of $\mathrm{Sym}(G)$. $\square$

2.3 Group actions

Since every group can be represented as a subgroup of a symmetric group, it is natural to draw parallels between certain group elements and certain permutations of a set. In particular, we can say that a group G acts on a set X through a particular type of homomorphism from G to $\mathrm{Sym}(X)$:

Definition 2.6. Let G be a group, and let X be a set. Then, a group action by G on X is a set of permutations $\pi_g : X \to X$ for all elements $g \in G$ such that

- The identity of G maps to the identity of $\mathrm{Sym}(X)$: If e is the identity element of G, $\pi_e(x) = x$ for all $x \in X$.
- For all elements $g_1, g_2 \in G$, $\pi_{g_1} \circ \pi_{g_2} = \pi_{g_1 g_2}$.

Example 2.5. Let G be a group. Then, G acts on itself (G acts on the set $X = G$) by left multiplication: For all $g, x \in G$, $\pi_g(x) = gx$.

It is sometimes said that a group acts on a set in 'the usual way.' This implies that there exists a natural group action between the two, coming from the group's construction. For example, the symmetric group S_n is given as the group of symmetries of a set of size n. The dihedral group $\mathbb{D}_n$ is given as the group of symmetries of a regular n-gon. Such wording is demonstrated below:

Example 2.6. Let G be the symmetric group S_n, and let X be the set $\{1, 2, \ldots, n\}$. Then, G acts on X the usual way, where $\pi_g(x) = g(x)$ for any $g \in G$, $x \in X$. Notice that every g is inherently a permutation of the elements of X.

In practice, we can refer to $\pi_g(x)$ as $g \cdot x$ or gx, bypassing the function notation. When using this notation, it is important to distinguish between multiplication of elements of the same group and permutation of X by g, especially when there are multiple elements present in a product, such as in $g_1 g_2 x_1 x_2 = \pi_{g_1 g_2}(x_1) \cdot x_2$.

Theorem 2.2. Let G be a group acting on a set X. Let $g \in G$ and $x \in X$ and $y = gx$. Then, $x = g^{-1}y$. Also, if $x' \in G$ such that $gx' = gx$, then $x' = x$.

Proof. We first prove the first half concerning x and y. Using $y = gx$, we have $g^{-1}y = g^{-1}(gx) = (g^{-1}g)x = ex = x$.

Now we show that $gx' = gx$ implies $x' = x$. Multiplying both sides of the first equation by g^{-1}, we have $g^{-1}(gx') = g^{-1}(gx)$, which leads to $(g^{-1}g)x' = (g^{-1}g)x$, so $ex' = ex$, so $x' = x$. Notice that this demonstrates the injectivity of multiplying by g(since g is a permutation of the set X). $\square$

We can alternatively define a group action as a homomorphism from G to the symmetric group $\mathrm{Sym}(X)$.

Theorem 2.3. Let G be a group and X be a set. A group action by G on X is the same as a homomorphism $f : G \to \mathrm{Sym}(X)$.

Proof. First we show that we can represent any given group action as a homomorphism $f : G \to \mathrm{Sym}(X)$.

To show that the mapping f to be a homomorphism, we must first show that f respects group operations. In particular, given any two elements $g_1, g_2 \in G$, $f(g_1) \circ f(g_2) = f(g_1 g_2)$. This follows directly from the group action definition: $\pi_{g_1} \circ \pi_{g_2} = \pi_{g_1 g_2}$, and $f(g) = \pi_g$ for all $g \in G$, so $f(g_1) \circ f(g_2) = f(g_1 g_2)$.

Notice that for f to be a homomorphism it must also map the identity of G to the identity of $\mathrm{Sym}(X)$: this also follows from the group action definition, which states that $\pi_e(x) = x$ for the identity $e \in G$, so $\pi_e = f(e)$ is the identity of $\mathrm{Sym}(x)$. Therefore f is a homomorphism.

Next we show that we can represent any homomorphism $f : G \to \mathrm{Sym}(X)$ as a group action.

Let $f(g) = \pi_g$ for all $g \in G$. Then $f(g_1) \circ f(g_2) = f(g_1 g_2)$, so $\pi_{g_1} \circ \pi_{g_2} = \pi_{g_1 g_2}$.

Additionally, for the identity element $e \in G$, $f(e)$ is the identity permutation, so $\pi_g(x) = x$ for all $x \in X$. Therefore G acts on X. $\square$

Example 2.7. Let $\mathbb{R}$ be the group of real numbers under addition. Then, R acts on itself by translation: for any two elements $p, q \in \mathbb{R}$, the mapping $\pi_p : \mathbb{R} \to \mathbb{R}$ is defined by $\pi_p(q) = p + q$. This satisfies $\pi_p \circ \pi_q = \pi_{p+q}$, and $\pi_0(q) = q$, so it is a group action.

Example 2.8. Let G be the group of moves of a Rubik's cube. Then, G acts on both the set of vertices and the set of edges of a Rubik's cube, since each move can be interpreted as a permutation of vertices and a permutation of edges.

Example 2.9. Let G be the group of possible positions of a Rubik's cube, with the identity being the solved cube. Then, each position can be interpreted as a permutation of the colors, so G acts on itself.

Notice that because of the formula $(gh)^{-1} = h^{-1}g^{-1}$, a left action can be constructed from a right action by composing with the inverse operation of the group. Therefore, sources may occasionally refer to the right multiplication group action in place of the left multiplication group action for a result that was previously proven true for the left multiplication group action.

2.4 Orbits and stabilizers

The structure of a group action can be represented in two parts:

Definition 2.7. Let group G act on a set X. The orbit $\mathrm{Orb}_x \subset X$ is the set of elements $x' \in X$ for which there exists a g such that $gx = x'$.

Definition 2.8. Let group G act on a set X, and let $x \in X$. The stabilizer $\mathrm{Stab}_x \subset G$ is the set of elements $g \in G$ such that $gx = x$.

Definition 2.9. Let group G act on a set X. An element $x \in X$ is a fixed point when $\mathrm{Stab}_x = G$, or, equivalently, when $\mathrm{Orb}_x = \{x\}$.

Example 2.10. The group of 2x2 matrices of real numbers $\mathrm{GL}_2(\mathbb{R})$ acts on $\mathbb{R}^2$ by multiplication. In this action, the stabilizer of $\mathbf{0} = (0, 0) \in \mathbb{R}^2$ is $GL_2(\mathbb{R})$, since any 2x2 matrix multiplied by $\mathbf{0}$ is $\mathbf{0}$. On the other hand, the orbit of $\mathbf{0}$ is $\{\mathbf{0}\} \in \mathrm{GL}_2(\mathbb{R})$.

Example 2.11. Notice that the Rubik's Cube group acts on the set of non-center face cubelets of the Rubik's Cube. This action creates two disjoint orbits: edge pieces and corner pieces.

Example 2.12. Let G be a group. Then, G acts on itself ($X = G$) by conjugation, in which $gx = gxg^{-1}$. In this action $x \in G$ is a fixed point if and only if

$$gxg^{-1} = x = xe = xgg^{-1}$$

for all $g \in G$, so x must commute with every element of G, so the fixed points of this action make up the center of G. Also notice that every orbit is a normal subgroup in G.

Definition 2.10. We say a group G acts transitively on a set X if there exists an element $x \in X$ such that $\mathrm{Orb}_x = X$.

Definition 2.11. We say a group G acts simply transitively on a set X if it acts transitively and that for any $g \in G, x \in X$, $gx = x$ if and only if $g = e$. In other words, Stab_x is trivial for all $x \in X$.

Theorem 2.4 (Fundamental Theorem of Group Actions)**.** Let G be a group acting on a set X.

- Different orbits of the action are disjoint and form a partition of X.
- For each $x \in X$, Stab_x is a subgroup of G and $\operatorname{Stab}_{gx} = g \operatorname{Stab}_x g^{-1}$.
- For each $x \in X$, there is a bijection $\operatorname{Orb}_x \to G/\operatorname{Stab}_x$ by $gx \mapsto g \operatorname{Stab}_x$. More concretely, $gx = g'x$ if and only if g and g' lie in the same left coset of Stab_x, and different left cosets of Stab_x correspond to different points in Orb_x.

From part (a) of the fundamental theorem we can see that a group action is transitive if and only if it has one orbit. In other words, $\operatorname{Orb}_x = X$ for some $x \in X$ if and only if $\operatorname{Orb}_x = X$ for all $x \in X$.

Stemming from part (c) of the fundamental theorem we have the orbit-stabilizer formula:

Theorem 2.5 (Orbit-Stabilizer Formula)**.** Let G be a group acting on a set X, and let $x \in X$. Then, the number of cosets of Stab_x equals the size of the orbit Orb_x:

$$|\operatorname{Orb}_x| = |G : \operatorname{Stab}_x|$$

3 Cayley Graphs

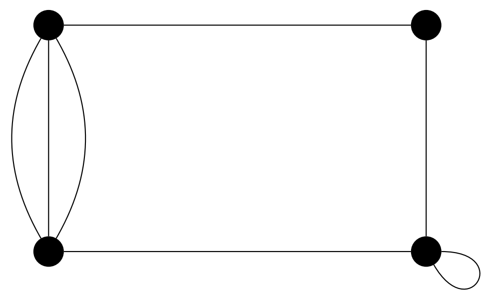
3.1 Graph theory basics

Now we familiarize ourselves with basic definitions from graph theory.

Definition 3.1. A graph $\Gamma = (V, E)$ is a set of vertices and edges, where each edge $e \in$ E is a connection between two not necessarily distinct vertices $v, w \in V$.

Definition 3.2. An undirected graph is a graph in which none of the edges are one-way: a directed graph or digraph, on the other hand, is a graph in which at least one of the edges is one-way.

Example 3.1. The following is an example of an undirected graph:



Notice that between the two vertices on the left there are 3 edges. Whenever there are 2 or more edges connecting the same pair of vertices, it is called a multi-edge or multiple edge.

Also, notice that an edge connects the bottom right vertex to itself. We call this a loop: an edge in which both endpoints are the same point.

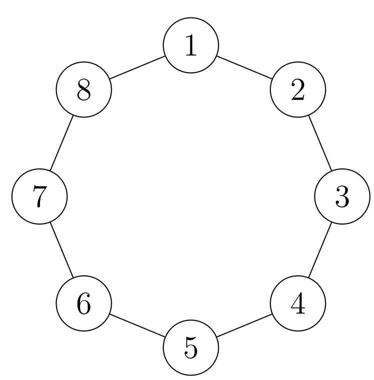
Now, notice that we can navigate from any vertex to any other vertex by traveling across a sequence of edges.

Definition 3.3. A path is a sequence of edges that connects one vertex of a graph to another vertex of a graph.

Definition 3.4. If there exists a path between every pair of vertices in a graph, the graph is connected.

In the last section we explored how groups act on sets. In fact, groups can also act on graphs, since graphs also have automorphism groups. For example,

Example 3.2. Consider the cycle graph C_8, shown below.



Notice that the automorphisms of C_8 take the form of rotations and reflections, described by the dihedral group $\mathbb{D}_8$, which is comprised of all rigid transformations of a regular octagon. Therefore, $\mathbb{D}_8$ acts on C_8.

3.2 Cayley graph basics

Now we define Cayley color diagrams, Cayley digraphs, and Cayley graphs.

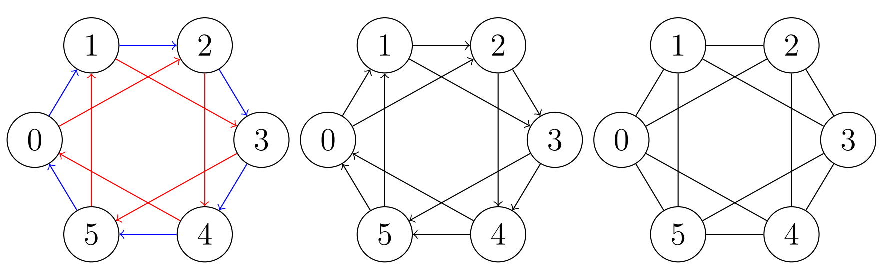
Definition 3.5. Let G be a group, and let $S \subseteq G$ be a generating set of G. The Cayley color diagram $\Gamma_c(G, S)$ is a colored connected digraph with one vertex corresponding to each element $g \in G$, one color c_s corresponding to each element $s \in S$, and one directed edge (g, gs) of color c_s for any $g \in G, s \in S$.

If we take the Cayley color diagram $\Gamma_c(G, S)$ and ignore colors, we obtain the Cayley digraph $\vec{\Gamma}(G, S)$.

If we take the Cayley digraph $\vec{\Gamma}(G, S)$ and ignore directions, we obtain the Cayley graph $\Gamma(G, S)$. A more formal definition is as follows:

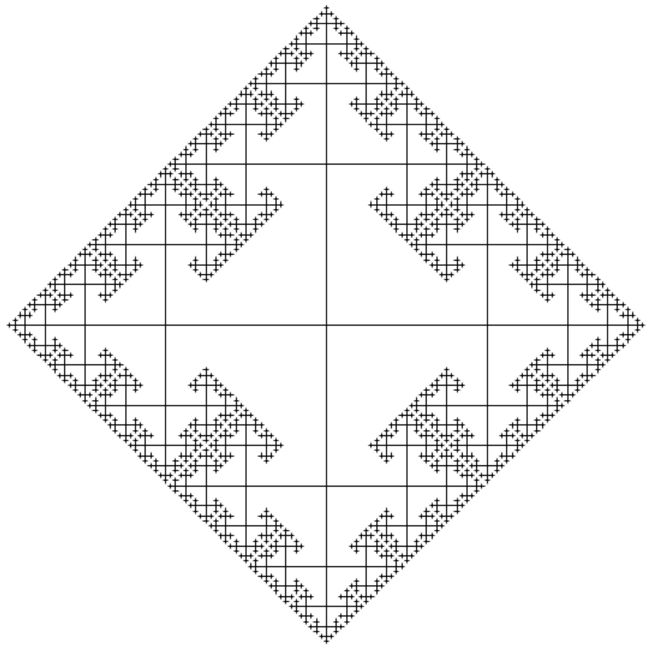
Definition 3.6. Let G be a group, and let $S \subseteq G$ be a generating set of G. The Cayley graph $\Gamma(G, S)$ is an uncolored, undirected graph with one vertex corresponding to each element $g \in G$, and one edge (g, h) for any $g, h \in G$ such that $gh^{-1} \in S$.

Example 3.3. Let G be the additive group of integers modulo 6, and let $S = \{1, 2\}$. Below are, from left to right, the Cayley color diagram $\Gamma_c(G, S)$, the Cayley digraph $\vec{\Gamma}(G, S)$, and the Cayley graph $\Gamma(G, S)$.

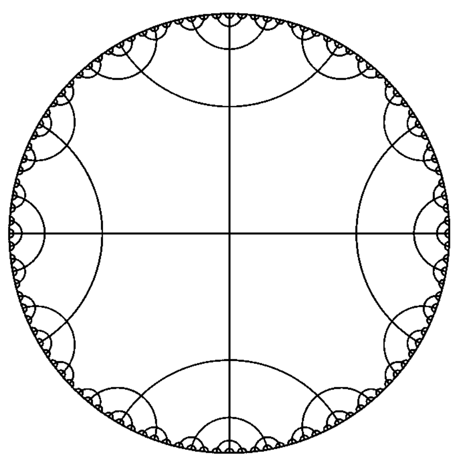


Example 3.4. Let F_2 be the free group of generating set of size 2. In other words, for two generators x and y, F_2 is the group of all products of x, y, x^{-1}, and y^{-1}, unique up to the group axioms($xx^{-1} = x^{-1}x = e$,$yy^{-1} = y^{-1}y = e$).

Then, a beautiful Cayley graph arises from the definition with $S = \{x, y\}$ as the generating set. In particular, the Cayley graph $\Gamma(F_2, S)$ is



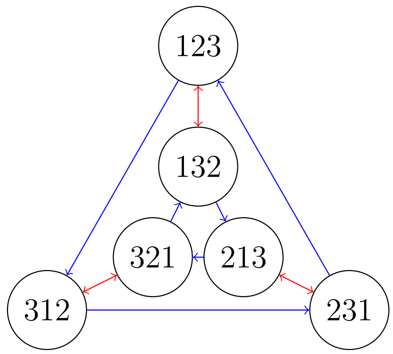
Alternatively, under the Poincaré hyperbolic disk model, we can visualize the same graph as

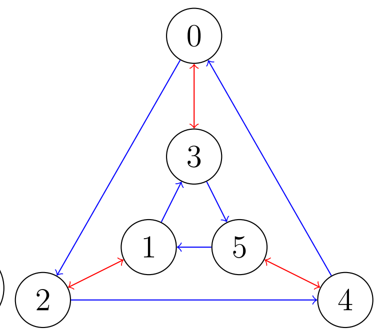


Example 3.5. In some cases, the same graph can be represented as a Cayley graph multiple ways using different groups. For instance, consider the symmetric group $G = S_3$. Notice that the subset $S = \{132, 312\} \subset S_3$ generates S_3: every non-identity element can be written as a product of 132 and 312.

Now consider the additive group modulo 6 $G' = \mathbb{Z}_6$. Notice that the subset $S' = \{2, 3\}$ generates $\mathbb{Z}_6$

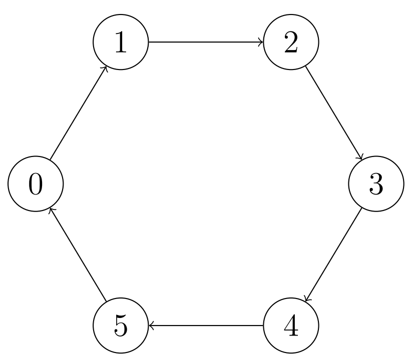
Now we construct both Cayley color diagrams. Below, from left to right, are the Cayley color diagrams $\Gamma_c(G, S)$ and $\Gamma_c(G', S')$:

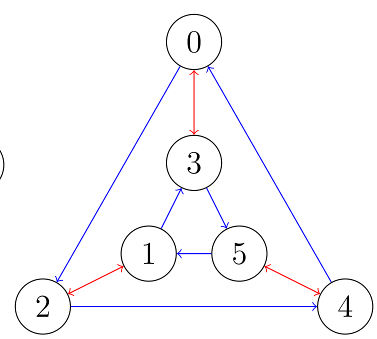




Notice that besides the labels on the vertices, the two graphs are identical in structure. Therefore, the same color diagram can be represented as a Cayley color diagram in at least two ways. This poses an interesting question which we will delve into in a later section: given a graph we know to be Cayley, how many ways can it be represented as a Cayley graph of distinct groups?

Example 3.6. The same group can generate multiple distinct Cayley graphs. Consider, for example, the additive group $G = \mathbb{Z}_6$. The subsets $S = \{1\}$ and $S' = \{2, 3\}$ both generate G, but they produce different Cayley graphs. Below are $\Gamma_c(G, S)$ and $\Gamma_c(G, S')$, respectively:





The aforementioned is the most commonly used definition in the context of showing that a given graph can be represented as a Cayley graph. However, some sources use the term "Cayley graph" to describe Cayley color diagrams and Cayley digraphs. To complicate things further, in some literature, S does not necessarily generate G, in which case the Cayley graph is not necessarily connected. Some literature also requires S to be closed under inversion for undirected Cayley graphs. The generating set S is also often assumed to not contain the identity element e of G.

3.3 General properties

In considering the automorphisms of Cayley graphs, we must first lay out some general properties of Cayley graphs:

Theorem 3.1. Let G be a group with generating set S. Every Cayley (di)graph $\Gamma(G, S)$ is regular.

Proof. Let s be an element of S such that $s^2 \neq e$. Then, for every vertex $g \in G$, there exists exactly two edges generated by s which connect to g: one edge between g and gs, and one edge between gs^{-1} and g. We show this by contradiction: suppose there exists a third distinct edge generated by s connecting g to an element h. Then, $hs = g$ or $gs = h$, so $h = gs$ or $h = gs^{-1}$, so this edge is not distinct.

Now, suppose $s \in S$ satisfies $s^2 = e$. Then, $gs^{-1} = gs$, so s generates exactly one edge connected to g for any $g \in G$.

Therefore, the degree of every vertex in a Cayley graph is the same, so $\Gamma(G, S)$ is regular.

In a Cayley digraph, the indegree of each vertex is equal to the outdegree, which is equal to the size of the generating set. Notice that in a Cayley digraph the edges generated by $s^2 = e$ can be treated as undirected edges, since there exists a directed edge going from g to gs and from gs to g. $\square$

Definition 3.7. A graph Γ is vertex-transitive if its automorphism group $\mathrm{Aut}(\Gamma)$ acts transitively on its set of vertices $\mathrm{V}(\Gamma)$.

Theorem 3.2. Let G be a group with generating set S. Every Cayley (di)graph $\Gamma(G, S)$ is vertex-transitive.

Proof. We first show that the left transformation $\ell_g (g \in G)$ is an isomorphism.

Let $a, b \in G$ be vertices in $\vec{\Gamma}(G, S)$ with an edge from a to b. Then, $as = b$ for some $s \in S$. Therefore $gas = gb$, so

$$\ell_g(a)s = \ell_g(b)$$

so the mapping ℓ_g preserves edges.

Conversely, let $a, b \in G$ be vertices in $\vec{\Gamma}(G, S)$ such that there is no edge from a to b. Therefore, $ac = b$ for some $c \notin S$. Therefore, $gac = gb$ for some $c \notin S$, so

$$\ell_g(a)c = \ell_g(b)$$

so the mapping ℓ_g preserves nonedges. Since ℓ_g preserves both edges and nonedges, it is a homomorphism.

Next we show that between any two elements $a, b \in G$ there exists an isomorphism ℓ_g such that $\ell_g(a) = b$. In particular, consider $g = ba^{-1}$. Then,

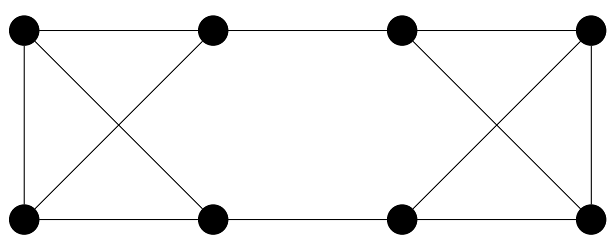
$$\ell_g(a) = ba^{-1}a = be = b$$

Therefore, there exists an automorphism mapping any given vertex to another given vertex, so $\vec{\Gamma}(G, S)$ is vertex-transitive.

In addition, since $\mathrm{Aut}(\vec{\Gamma}(G, S)) \leq \mathrm{Aut}(\Gamma(G, S))$, the undirected Cayley graph $\Gamma(G, S)$ is also vertex-transitive. □

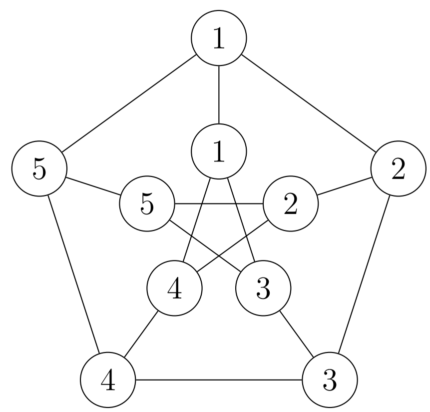
Notice that all vertex-transitive graphs are regular, since each vertex must have identical structure for a graph to be vertex-transitive. However, not all regular connected graphs are vertex-transitive.

Example 3.7. Consider the following graph:



This is the smallest regular, non-vertex-transitive graph. In particular, if we look at the bottom left vertex and the vertex directly to the right of it, we see that there is no automorphism mapping one to the other. Therefore, in the action by the automorphism group on the vertices there exists no orbit containing both vertices, so there must be at least two orbits, so the graph is non-vertex-transitive.

Example 3.8. In addition, not all vertex-transitive graphs are Cayley graphs. The Petersen graph is the smallest vertex-transitive graph which cannot be represented as a Cayley graph.



The proof idea utilizes the fact that there are only two distinct groups of size 10: the dihedral group $\mathbb{D}_5$ and the cyclic group $\mathbb{Z}_{10}$. In addition, at least one element $s \in S$ must have order 2 ($s^2 = 1$), since the degree of each vertex is odd. Then we do casework on the two groups to show that no possible Cayley graph has a 5-cycle but no 4-cycle like the Petersen graph does.

Lemma 3.3. Let G be a group with generating set S. Let $\ell(G)$ be the image of the left multiplication group action by G on itself. Then, $\ell G = \mathrm{Aut}(\Gamma_c(G,S)) \subseteq \mathrm{Aut}(\Gamma(G,S))$.

Proof. Let $s = s_1^{a_1} \cdots s_n^{a_n}$ where n is an integer, $s_1, \ldots, s_n \in S$, and $a_i = \pm 1$. Then, let ϕ be an automorphism in $\mathrm{Aut}(\Gamma_c(G,S))$ such that $\phi(e) = g$ for some $g \in G$. Then,

$$\phi(es) = \phi(es_1^{a_1} \cdots s_n^{a_n}) = \phi(e)s_1^{a_1} \cdots s_n^{a_n} = gs$$

Also notice that every element of G can be written as $s_1^{a_1} \cdots s_n^{a_n}$, so $\phi = \ell_g$ for all $g \in G$. Therefore every element of $\mathrm{Aut}(\Gamma_c(G,S))$ can be represented as a left translation.

Also, as shown above in the proof of vertex-transitivity, $\ell_g \in \mathrm{Aut}(\Gamma_c(G,S))$ for all $g \in G$. Therefore $\ell G = \mathrm{Aut}(\Gamma_c(G,S))$. $\mathrm{Aut}(\Gamma_c(G,S)) \subseteq \mathrm{Aut}(\Gamma(G,S))$ is by definition. $\square$

Theorem 3.4. A connected graph $\Gamma = (V,E)$ is Cayley if and only if there exists a subgroup $H \subseteq \mathrm{Aut}(\Gamma)$ which acts simply transitively on V.

Proof. The only if direction is clear: if Γ is Cayley, there exists a group G (whose elements are just V) and generating set S such that $\Gamma = \Gamma(G,S)$.

The image of the left multiplication action $\ell G = H \subseteq \mathrm{Aut}(\Gamma)$ acts simply transitively on V (or G).

Now we prove the other direction. Let $\Gamma = (V, E)$ be a connected graph with $H \subseteq \mathrm{Aut}(\Gamma)$ acting simply transitively on V. In another word, for all $v \in V$, Stab_v are trivial. Therefore, $|H| = |V|$. Fix a vertex $v \in V$ and let S be the set of all elements $h \in H$ such that there exists an edge $\{v, hv\} \in E$. This is symmetric since H acts by graph automorphisms, so that $\{v, hv\} \in E$ if and only if $\{h^{-1}v, v\} \in E$. We now define the isomorphism between Γ and $\Gamma(H, S)$ in the following way: for $u \in V$, there exists a unique $h \in H$ such that $hv = u$, so map $u \mapsto h$; therefore, edge $(u, v) \in E$ is included in $\Gamma(H, S)$ by h according to the definition of S. □

Equivalently, the result is also stated as follows.

Theorem 3.5 (Sabidussi [1964]). A connected graph $\Gamma = (G, E)$ is a Cayley graph of the group G if and only if $\ell G \leq \mathrm{Aut}(\Gamma)$.

3.4 Vertex-transitive graphs

We know that all Cayley graphs are vertex-transitive. Now, we focus on figuring out which specific vertex-transitive graphs are Cayley or not.

The Kneser's graph $KG_{n,k}$ $(n \geq 2, k \geq 1)$ has $\binom{2n+k}{n}$ vertices identified with the set of n-tuples of a $(2n + k)$-set; two vertices are adjacent if the corresponding n-tuples are disjoint. Petersen's graph is $KG_{2,1}$.

Theorem 3.6 (Kantor [1972],Godsil [1980]).

(a) Kneser's graph $KG_{n,k}$ $(n \geq 2, k \geq 1)$ is a Cayley graph precisely if $n = 2$ and $2n + k$ is a prime power, $2n + k \equiv -1 \pmod 4$, or $n = 3$ and $2n + k \in \{8, 32\}$

(b) If $n \geq 4$ then, with some exceptions, the only transitive proper subgroup of $\mathrm{Aut}(KG_{n,k})$ is the one induced by the alternating group A_{2n+k}. Exceptions occur for $n = 5$ when $2n + k \in 12, 24$ and for $n = 4$ when $2n + k \in \{9, 11, 12, 23, 24, 33\}$.

Theorem 3.7.

(a) (Wielandt [1964]) If G is a transitive group of degree p^k, where p is prime, then the Sylow p-subgroups of G are transitive as well.

(b) (Marušič [1985]) Every vertex-transitive (di)graph of order p^k, where k is an integer ≤ 3, is a Cayley (di)graph. For $k \geq 4$, counterexamples exist.

4 Cayley Representation Problems

Often, different groups and generating sets can result in identical Cayley graphs. The pair (G, S) is called a Cayley representation of a graph Γ if $\Gamma \cong \Gamma(G, S)$.

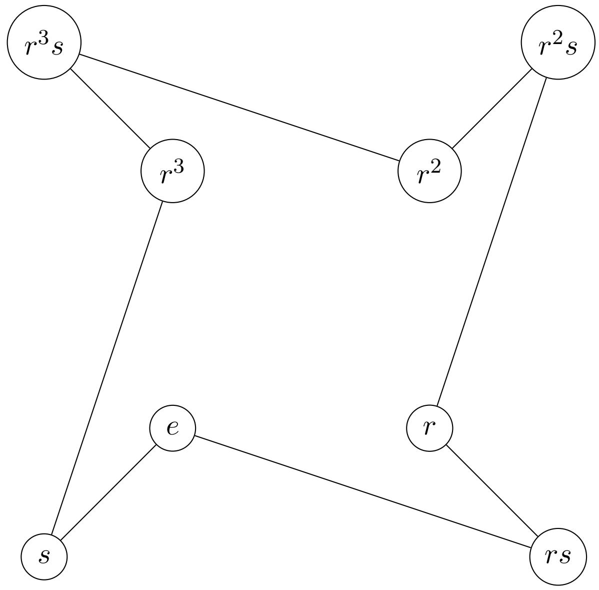
Given a Cayley graph Γ, there can be multiple different Cayley representations (G, S). For example, let $\Gamma = K_n$, the complete graph of order n. Any group G of order n and $S = G \setminus \{e\}$ results $\Gamma(G, S) \cong K_n$, and the number of such representations equals the number of non-isomorphic groups of order n.

The Cayley representation problem is: given a Cayley graph Γ, determine all Cayley representations (G, S). The above example shows the answer for complete graphs K_n.

Next, we examine complete d-partite graphs $K_{m;d}$, where there are d parts each with size m, a vertex connects to no vertexes in the same part and all vertexes in other parts.

Theorem 4.1 (Li [2002]). A group G has a Cayley graph isomorphic to $K_{m;d}$ if and only if G has order md and has a subgroup G_0 of order m, and the generating set is $G \setminus G_0$.

On the other hand, C_n $(n \geq 3)$, the cycle graph of size n is a 2-regular graph, the simplest type of regular graph. It follows from the definition of the dihedral group that the automorphism group $\mathrm{Aut}(C_n) = \mathbb{D}_n$ is the dihedral group. Now, notice that $\Gamma(\mathbb{Z}_n, \{a\}) \cong C_n$ for any $a \in \mathbb{Z}_n$ coprime to n. Also, note that $\mathbb{Z}_n$ is a subgroup of $\mathbb{D}_n$ with the order n. If n is odd, it is the only subgroup of order n. If n is even, $(\mathbb{D}_{\frac{n}{2}}, \{s, rs\})$ is also an Cayley representation for C_n, where r, s generates $\mathbb{D}_{\frac{n}{2}}$ with $r^{\frac{n}{2}} = s^2 = (rs)^2 = e$. The case of $\mathbb{D}_4$ is illustrated below, resulting in C_8. Note $\mathbb{D}_{\frac{n}{2}}$ is another subgroup of $\mathbb{D}_n$ of the order n. In fact, the above discussion also applies to C_2 even though it only has one edge, and is thus a 1-regular graph. It follows from Theorem 3.5 theses are the only Cayley representations for C_n.

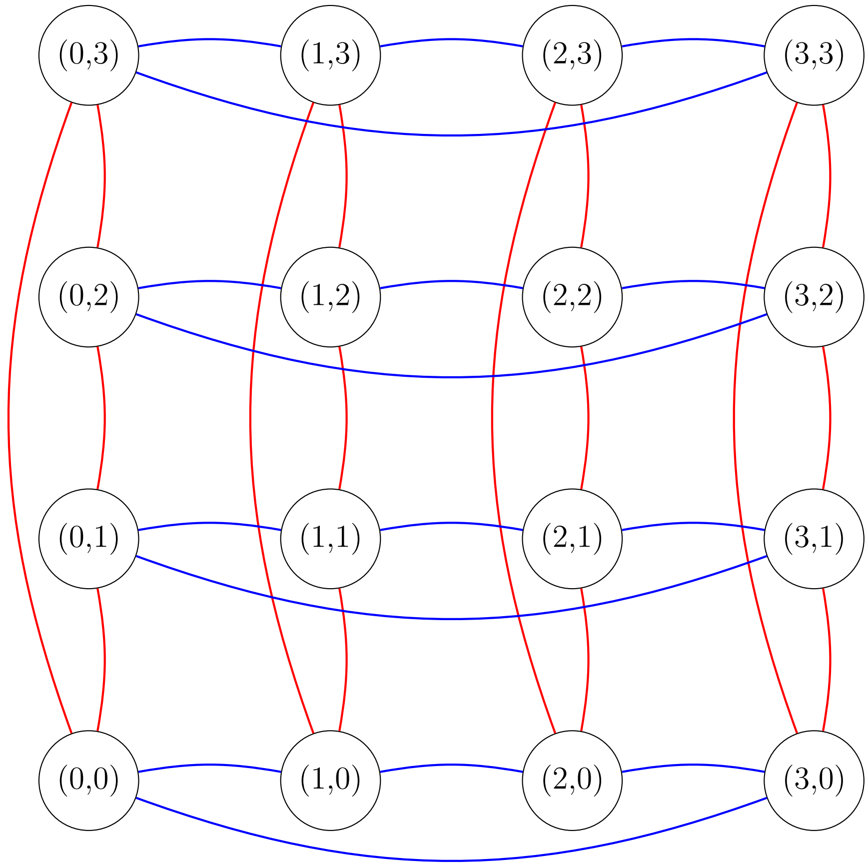


The next example is a generalization of the above and concerns the Cartesian product of cycles of the same length.

Definition 4.1. For given graphs $\Gamma_1 = (V_1, E_1)$ and $\Gamma_2 = (V_2, E_2)$, the Cartesian product $\Gamma_1 \times \Gamma_2$ is the graph with the vertex set $V_1 \times V_2$, where each element is a pair of elements from Γ_1 and Γ_2, and the edge set

$$\{\{(u_1, v_2), (v_1, v_2\}|\{u_1, v_1\} \in E_1, v_2 \in V_2\} \cup \{\{(u_1, u_2), (u_1, v_2\}|u_1 \in E_1, \{u_2, v_2\} \in V_2\}.$$

Example 4.1. Consider the cycle graph C_4. The Cartesian product of C_4 by itself, $C_4 \times C_4 = C_4^2$, is shown below:



We have the following results about possible groups G for these Cartesian products of cycle graphs.

Theorem 4.2 (Li [2002]). Let $\Gamma \cong C_n^d$, the Cartesian product of d copies of C_n. If $n = 3$ or $n \geq 5$,

- If n is odd, then Γ is a Cayley graph of a group $G \cong \mathbb{Z}_n^d$.
- If n is even, then Γ is a Cayley graph of a group $G \cong \prod_{i=1}^d G_d$, where $G_i \cong \mathbb{Z}_n$ or $\mathbb{D}_{\frac{n}{2}}$.

If $n = 2$, the graph is also known as Q_d, the hypercube of dimension d. In fact, the $n = 4$ case also boils down to a hypercube $C_4^d = C_2^{2d} = Q_{2d}$. For $d = 2$, the 2-dimensional hypercube Q_2 is identical to the square C_4, which we have already examined.

Theorem 4.3 (Li [2002])**.** Let $\Gamma \cong Q_d$, the d dimensional hypercube graph.

- If $d = 3$, Q_3 has these Cayley representations: $(\mathbb{Z}_2^3, \{(1,0,0),(0,1,0),(0,0,1)\})$; $(\mathbb{Z}_4 \times \mathbb{Z}_2, \{(i,0),(0,1)\})$, $i = 1$ or 3; or $(\mathbb{D}_4, \{r, s\})$ where r, s generates $\mathbb{D}_4$ with $r^4 = s^2 = (rs)^2 = e$.

- For general d, the size of Q_d is 2^d. For G of order 2^d and $G \cong \prod_{i=1}^{l} G_l$, where $G_i \cong \mathbb{Z}_2, \mathbb{Z}_4$ or $\mathbb{D}_4$, there exists $S \subset G$, such that $\Gamma(S, G) \cong Q_d$.

The complete answer for general Q_d is unknown. Dixon [1997] showed results for small dimensions $d = 3, 4, 5, 6$, there are 4,14,45 and 238 Cayley representations, respectively.

There are lots of open questions for the Cayley representation problem. Known results and open questions are surveyed in Li [2002], on which this section is based.

Acknowledgements

The author would like to thank his mentor, Nitya Mani, as well as Euler Circle and Simon-Rubinstein Salzedo for making this project possible.

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