Machines, Trees, and the Burnside Problem

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Euler Circle

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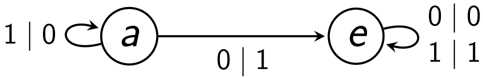
The Burnside Problem

The general Burnside problem [1902]. If a group is finitely generated, and each element is of finite order, is the group necessarily finite?

Mealy Machines

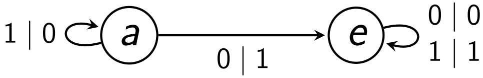


Input a *word* at some state, traverse the graph and build an output. Stop when the input word is consumed.

Instruction format: input | output

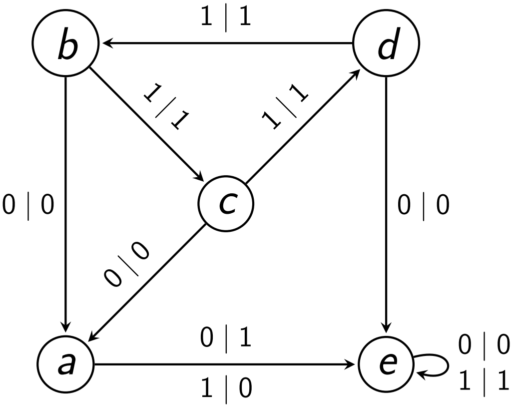
Example:
$a(0010) = e(001) + \text{“1”} = e(00) + \text{“11”} = e(0) + \text{“011”} = \text{“0011”}$
$a(1101) = \text{“1110”}$

Constructing a Group

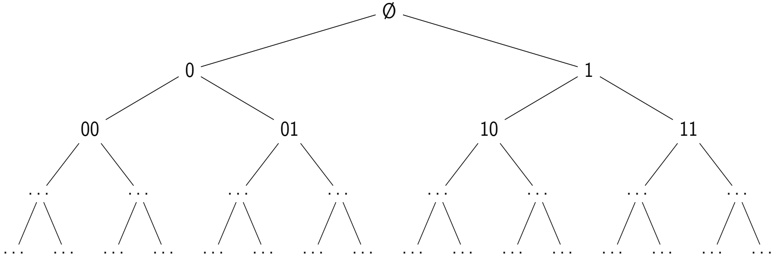


Elements of a group under function composition.
e is the identity, so what would something like a^2 be?
$a^2(w) = aa(w) = a(a(w)) = a(w+1) = w+2$
Only works as a group if we limit the word length/size l
Isomorphic to the cyclic group C_n where $n = 2^l$

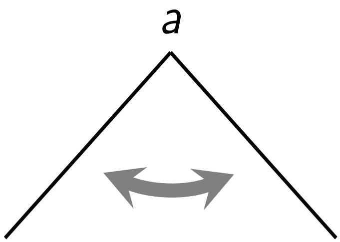
Grigorchuk Automata

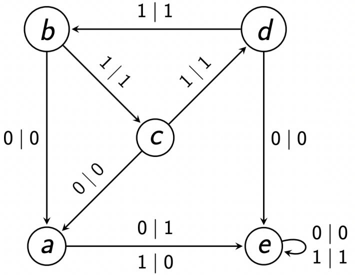


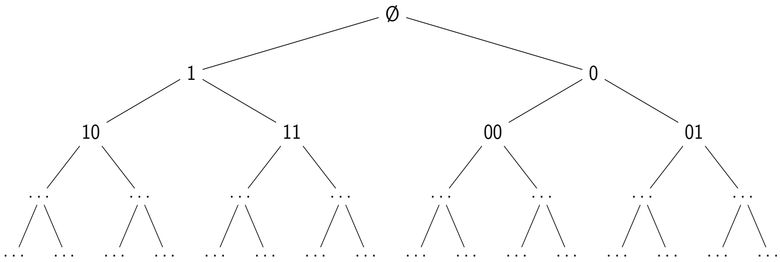
Binary Trees



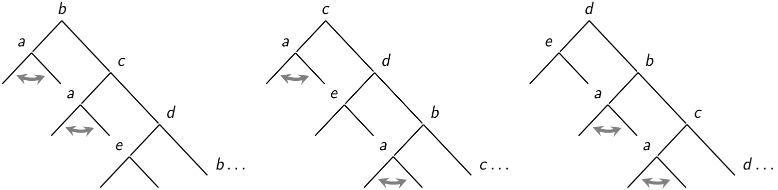
Transformations on Trees



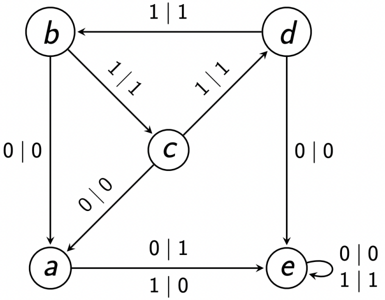




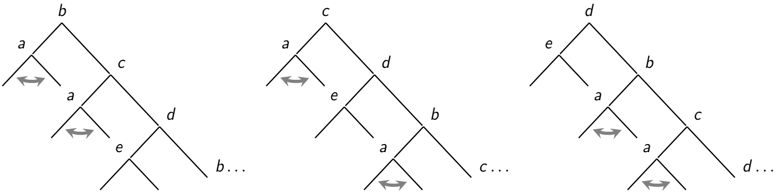
Other Transformation Trees



$$bc = cb = d$$
$$bd = db = c$$
$$cd = dc = b$$

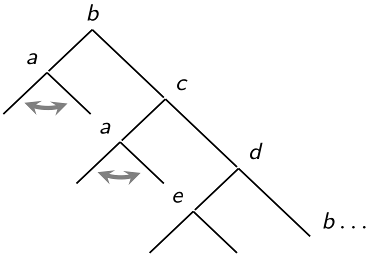


Breaking Down a Transformation (ψ)



$$\psi(b) = (a, c)$$
$$\psi(c) = (a, d)$$
$$\psi(d) = (e, b)$$

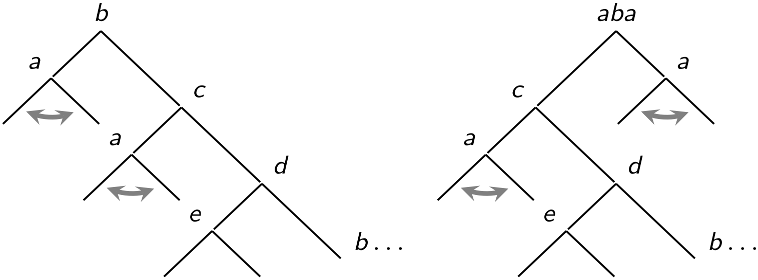
Properties of ψ



for $g_1 g_2 = s \in S$, $\psi(s) = \psi(g_1)\psi(g_2)$
$\psi(bc) = \psi(d) = (e, b)$
$\psi(b)\psi(c) = (a, c)(a, d) = (aa, cd) = (e, b)$

Conjugates and ψ

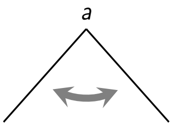
The conjugate of g by h is defined as hgh^{-1}
$a^2 = e \implies a^{-1} = a$
a conjugate of some g by a is $aga^{-1} = aga$

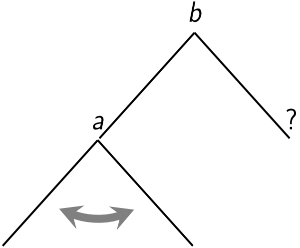


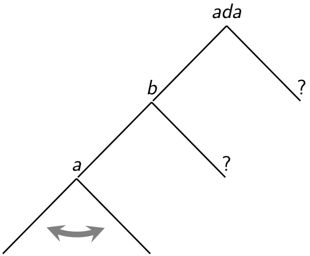
$\psi(aba) = (c, a)$
$\psi(aca) = (d, a)$
$\psi(ada) = (b, e)$

Generate an Infinite Number of Elements

Represent elements as words from the alphabet of the generating set $\{a, b, c, d, e\}$
For any word w, I can find some w' where $\psi(w')[0] = w$
For that w', I can find some w'' where $\psi(w'')[0] = w'$
For that w'', I can find some w''' where $\psi(w''')[0] = w''$
etc.


a


b
a
?

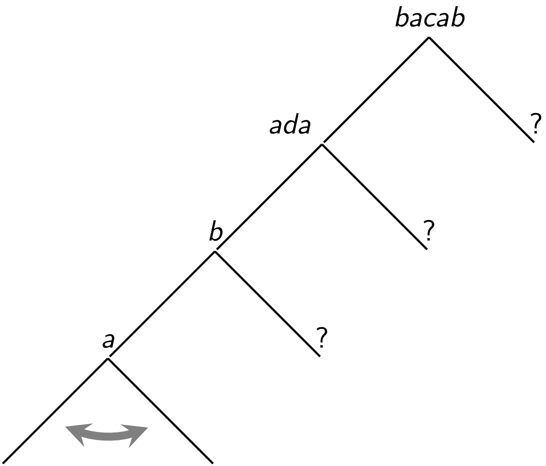

ada
b
?
a
?

uh oh

$$\psi(b) = (a, c)$$
$$\psi(c) = (a, d)$$
$$\psi(d) = (e, b)$$
$$\psi(aba) = (c, a)$$
$$\psi(aca) = (d, a)$$
$$\psi(ada) = (b, e)$$

what now?

we want *ada* to go to the left subtree, meaning $\psi(w) = (ada, ?)$.
$\psi(g_1 g_2) = \psi(g_1)\psi(g_2)$
Break w into w_1, w_2, w_3
$\psi(w) = \psi(w_1 w_2 w_3) = \psi(w_1)\psi(w_2)\psi(w_3) = (a, ?)(d, ?)(a, ?)$
$w_1 = b$
$w_2 = aca$
$w_3 = b$
so $w = b(aca)b$


bacab
ada
?
b
?
a
?

more

$(ada)b(aba)b(ada)$ applies $bacab$ to the left subtree

$b(aca)b(ada)b(ada)b(ada)b(aca)b$ applies $adabababada$ to the left subtree

bacabadabadabadabacab
adabababadabacabadabacabadabacabadabababada

Elements are Periodic

Long, semi-technical proof
Uses conjugates a LOT (order property)
Induction, 2 cases, one of which has 2 subcases, one of which has 3 subcases. Total of 5 cases/subcases.

The End

Thank you for listening!