# Random Graphs

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Euler Circle

July 2021

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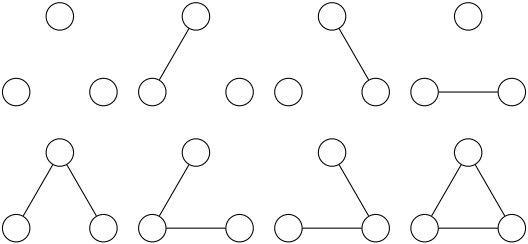

# Thresholds

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A threshold $r(n)$ for a property $A$ is a function such that, as $n \to \infty$,

$$\Pr[G(n,p) \in A] \to \begin{cases} 0 & \text{if } p \ll r(n), \\ 1 & \text{if } p \gg r(n), \end{cases}$$

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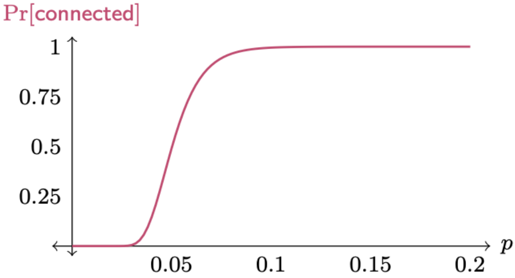

# History

- 1959: Erdős and Rényi published *On random graphs*, introducing the model $G(n, m)$
- 1959: Gilbert proposed $G(n, p)$ model
- 1969: Travers and Milgram studied the small world problem
- 1984: Béla Bollobás published *Random Graphs*
- 2006: Kahn-Kalai conjectured expectation threshold $\approx$ phase transition
- 2022: Park and Pham proved the Kahn-Kalai conjecture

# First and Second Moment Methods

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**Corollary**

*For nonnegative $X$,*

$$\Pr[X = 0] \leq \frac{\mathrm{Var}[X]}{\mathbb{E}[X]^2} = \frac{\mathbb{E}[X^2] - \mathbb{E}[X]^2}{\mathbb{E}[X]^2}.$$

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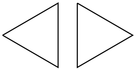
  - 4 cases: share 0,1,2,3 vertices

# Containing Triangles: Casework

- 0 shared vertices: $O(n^6)p^6$



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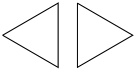

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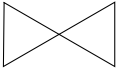

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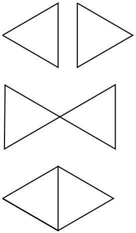

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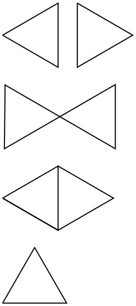

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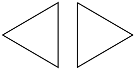

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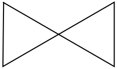

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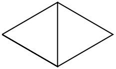

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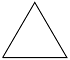

$$
\begin{aligned}
\mathrm{Var}[X] &= \mathbb{E}[X^2] - \mathbb{E}[X]^2 \\
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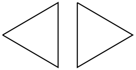

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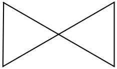

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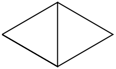

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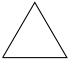

$$\begin{aligned}\mathrm{Var}[X] &= \mathbb{E}[X^2] - \mathbb{E}[X]^2 \\ &\sim O(n^6)p^6 + O(n^5)p^6 + O(n^4)p^5 + O(n^3)p^3 - O(n^6)p^6 \\ &= O(n^6)p^6\left(\frac{1}{n} + \frac{1}{n^2p} + \frac{1}{n^3p^3}\right)\end{aligned}$$

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**Theorem**

*The threshold for containing the complete graph on* $r$ *vertices* ($K_r$) *is* $p = n^{-\frac{2}{r-1}}$.

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  - Markov's Inequality: $\sum_{k=1}^{\lfloor n/2 \rfloor} \mathbb{E}[X_k] \to 0$ suffices

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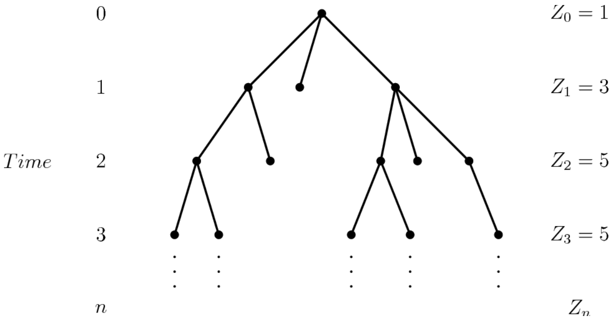

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  - $c > 1$ : giant component and small components

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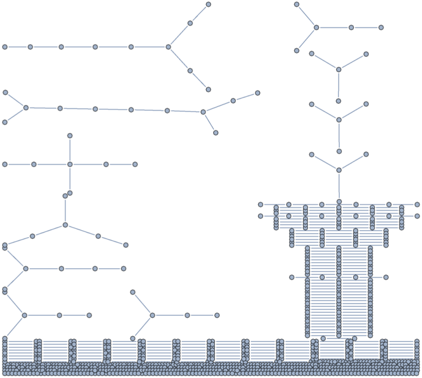

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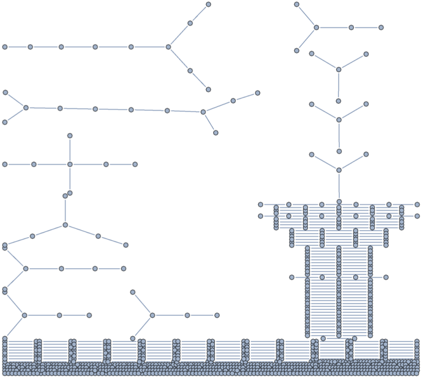

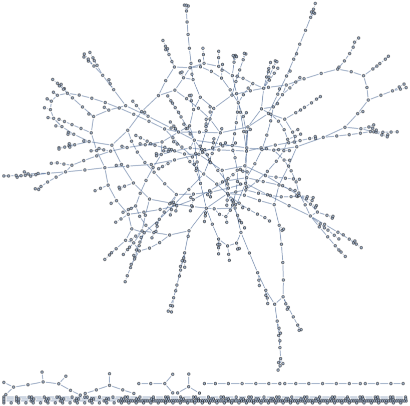

# Thank you

Thanks for listening!