Model theory in Tarski's geometry

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Logic symbols

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Also, need to define the symbol $\vDash$. Suppose we have a set of statements S and a statement C. Then $S \vDash C$ means that whenever S is true, C should be true.

Definition

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A *language* $\mathcal{L}$ is a set of symbols for functions, relations, constants.

Sentences

Definition

A *$\mathcal{L}$-sentence* is an $\mathcal{L}$-formula all variables of which have quantifiers $(\exists, \forall)$ before them.

Example

- $\forall y \exists x (x^2 = y)$ is a sentence
- $\exists x (x^2 = y)$ is not a sentence, because y does not have a quantifier

Theory and models

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Definition

We say that $\mathcal{M}$ is a *model* of a theory T (written $\mathcal{M} \models T$) iff $\mathcal{M} \models \phi$ for any $\phi \in T$.

So, a structure $\mathcal{M}$ can have a theory T. Then for theory T, $\mathcal{M}$ is a model. A theory T is called *satisfiable* if T has a model.

Model theory and axioms

Model theory is the study of the relationship between theories, and their models.

Very often, we have a class of structures in mind and try to write a set of properties T (i.e. a theory) describing these structures. We call these sentences of T *axioms*.

Proof

Definition

Proof of ϕ is a series of formulas such that the last formula is ϕ, and any formula in the series is either an axiom or follows from previous formulas by simple logical rules.

Definition

We write $T \vdash \phi$ if there is a proof of ϕ from T.

Theorem (Gödel's Completeness theorem)

Let T be an $\mathcal{L}$-theory and ϕ an $\mathcal{L}$-sentence, then $T \vdash \phi$ iff $T \vDash \phi$.

One direction of the theorem seems intuitively true because:

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Definition

We say that a theory T is *complete* if for any $\mathcal{L}$-sentence ϕ either $T \vDash \phi$ or $T \vDash \neg\phi$.

Definition

A language $\mathcal{L}$ is called *recursive* if there exists an algorithm that can check whether an arbitrary sequence of symbols is an $\mathcal{L}$-formula. A theory T called *recursive* if there exists an alorithm that can check whether a given sentence belongs to T.

We have an interesting theorem connecting completeness and recursiveness.

Theorem

If T is a recursive complete satisfiable theory in a recursive language $\mathcal{L}$, then T is decidable. That is there is an algorithm that when given an $\mathcal{L}$-sentence ϕ as input decides whether $T \models \phi$.

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Example

Let $\phi(a, b, c)$ be the formula

$$\exists x \; ax^2 + bx + c = 0.$$

Then by the quadratic formula,

$$\mathbb{R} \vDash \phi \leftrightarrow [(a \neq 0 \wedge b^2 - 4ac \geq 0) \vee (a = 0 \wedge (b \neq 0 \vee c = 0)].$$

Real closed fields

Definition

A *real closed field* F is a field that satisfies the following properties:

- There is a total order on F (i.e. any two elements are comparable)
- Every positive element of F has a square root in F
- Any polynomial of odd degree with coefficients in F has at least one root in F.

The set of axioms for real closed fields is commonly called RCF theory.

Theorem

The theory RCF admits elimination of quantifiers in the language of ordered rings $\mathcal{L}_{or} = \{+, -, \cdot, 0, 1, <\}$.

Elimination of quantifiers in RCF

Theorem

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Corollary

RCF is complete and decidable. Thus, RCF is the theory of $(\mathbb{R}, +, \cdot, <)$ *and decidable.*

Tarski's system of geometry

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(called elementary Euclidean geometry).

Figure: Alfred Tarski

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In 1920s a Polish-American mathematician, Alfred Tarski devised his own system of geometry, which contains a substantial fragment of Euclidean geometry
(called elementary Euclidean geometry).

After the development of this system, he proved that it is a model of the theory of real closed fields (RCF). Therefore, his system of geometry turned out to be decidable.

Figure: Alfred Tarski

Fundamental relations.

$B(x, y, z)$ - *betweenness* relation. It means that points x, y, z lie on a line and y is between x and z. $wx \equiv yz$ - *congruence* relation. It means that length of wx equals to the length of yz.

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Example

Initially Tarski proposed 24 axioms, but later their number was reduced to 11. Some axioms of Tarski's geometry:

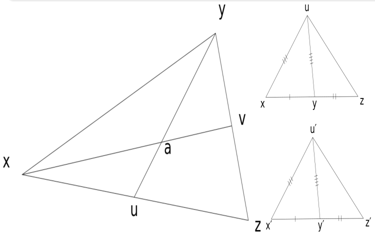
Identity of Congruence: $xy \equiv zz \to x = y$.
Transitivity of Congruence: $(xy \equiv zu \wedge xy \equiv vw) \to zu \equiv vw$.

Example

Some other examples:

Axiom of Pasch: $(Bxuz \wedge Byvz) \to \exists a\,(Buay \wedge Bvax)$.

Five Segment axiom: $(x \neq y \wedge Bxyz \wedge Bx'y'z' \wedge xy \equiv x'y' \wedge yz \equiv y'z' \wedge xu \equiv x'u' \wedge yu \equiv y'u') \to zu \equiv z'u'$.



From geometry to algebra

Lemma

All axioms of Tarski's system of plane geometry can be restated in formulas of RCF.

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Proof. Note that we can assign to a point x the coordinates (x_1, x_2), then. Betweenness:

$$B(x,y,z) \leftrightarrow [(x_1 - y_1)\cdot(y_2 - z_2) = (x_2 - y_2)\cdot(y_1 - z_1)] \wedge \\ \wedge [(0 \le (x_1 - y_1)\cdot(y_1 - z_1))] \wedge [0 \le (x_2 - y_2)\cdot(y_2 - z_2)]$$

Equivalence:

$$xy \equiv zu \leftrightarrow (x_1 - y_1)^2 + (x_2 - y_2)^2 = (z_1 - u_1)^2 + (z_2 - u_2)^2.$$

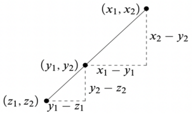

FIGURE 21. The definition of betweenness in the Cartesian plane.

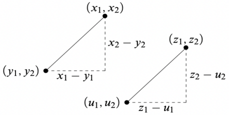

FIGURE 22. The definition of equidistance in the Cartesian plane.

Thanks for your attention!