Complex Lie Algebras

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Lie Algebra = Lie Group?

Definition

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Definition

A Lie algebra is the tangent space to a Lie group at the identity.

1. Lie algebra for every Lie group
2. Lie group for (almost) every Lie algebra!

Theorem (Lie's Third)

For each Lie algebra $\mathfrak{g}$ over $\mathbb{R}$, there is an associated Lie group G.

Visual Lie Algebra

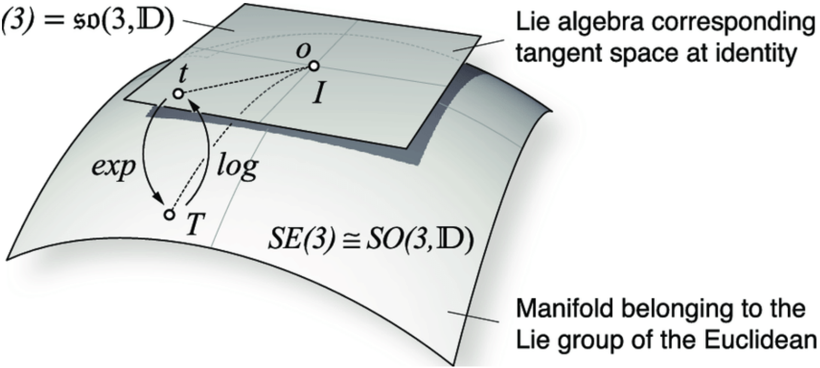

Figure. Lie Algebra of a Lie Group

Lie Algebras

Definition

A Lie algebra $\mathfrak{g}$ over a commutative field $\mathbb{F}$ is a vector space equipped with an operation $[.,.] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ for which the following axioms hold:

1. The operation $[.,.]$ is bilinear.
2. For all $x \in \mathfrak{g}, [x,x] = 0$.
3. For all $x, y \in \mathfrak{g}, [x,y] = -[y,x]$.
4. For all $x, y, z \in \mathfrak{g}, [x,[y,z]] + [y,[z,x]] + [z,[x,y]] = 0$.

General Linear Algebra

Example

The general Lie algebra $\mathfrak{gl}(V)$ is the vector space of all endomorphisms of V (linear maps from V to itself), with Lie bracket $[x, y] = xy - yx$ for all $x, y \in \mathfrak{gl}(V)$.

Lie Subalgebra

Definition

A Lie subalgebra $\mathfrak{h}$ of a Lie algebra $\mathfrak{g}$ is a vector subspace of $\mathfrak{g}$ where for all $x, y \in \mathfrak{h}, [x, y] \in \mathfrak{h}$.

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Example

Another lie algebra is the algebra $\mathfrak{sl}(2, \mathbb{C})$, the special linear algebra of dimension 2. This is defined to be all two dimensional matrices over $\mathbb{C}$ containing trace 0. To show that this is a subalgebra of $\mathfrak{gl}(2, \mathbb{C})$, we have

$$\begin{aligned} \mathrm{Tr}([x, y]) &= \mathrm{Tr}(xy - yx) \\ &= \mathrm{Tr}(xy) - \mathrm{Tr}(yx) \\ &= 0. \end{aligned}$$

Ideals

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Definition

A Lie algebra $\mathfrak{g}$ is simple if it does not have any ideals besides 0 and $\mathfrak{g}$. Additionally, $\mathfrak{g}$ cannot be commutatitve, ie; it cannot be true that $[x, y] = 0$ for all $x, y \in \mathfrak{g}$.

Homomorphisms

Definition

A linear transformation $\phi : \mathfrak{g} \to \mathfrak{h}$ between two Lie algebras is said to be a Lie homomorphism if

$$\phi([x, y]) = [\phi(x), \phi(y)]$$

for all $x, y \in \mathfrak{g}$.

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Definition

A representation of a Lie algebra is a Lie homomorphism $\phi : \mathfrak{g} \to \mathfrak{gl}(V)$ where both $\mathfrak{g}$ and V are vector spaces over $\mathbb{F}$.

Reducible Representations

Definition

For a representation of a Lie algebra $\phi : \mathfrak{g} \to \mathfrak{gl}(V)$, a subspace W of V is invariant if $\phi(x)w \in W$ for $w \in W$ and $x \in \mathfrak{g}$.

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Definition

A Lie algbera is irreducible if its only invariant subspaces are V and the zero space.

Theorem (Weyl)

For a semisimple Lie algebra $\mathfrak{g}$ and a finite representation ϕ, ϕ is the direct sum of irreducible representations.

Adjoints

Definition

The adjoint of an element x in a Lie algebra $\mathfrak{g}$ is the map $\operatorname{ad} x : \mathfrak{g} \to \mathfrak{g}$ defined by $\operatorname{ad} x(y) = [x, y]$ for all $y \in \mathfrak{g}$.

Definition

The adjoint representation of a Lie algebra is the map that sends each element of a Lie algebra to its adjoint. In other words, the adjoint representation is the map $\operatorname{ad} : \mathfrak{g} \to \mathfrak{gl}(V)$ defined by $\operatorname{ad}(x) = \operatorname{ad} x$ for all $x \in \mathfrak{g}$. The image of ad is denoted as $\operatorname{ad}(\mathfrak{g})$.

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Definition

If $\mathfrak{g}$ is semisimple, its roots are the nonzero eigenvalues of its adjoint representation.

Nilpotency

Definition

The descending central series of a Lie algebra $\mathfrak{g}$ is the series

$$\mathfrak{g}^0 = \mathfrak{g}, \ldots, \mathfrak{g}^i == [\mathfrak{g}, \mathfrak{g}^{i-1}].$$

A lie algebra $\mathfrak{g}$ is nilpotent if there exists some n such that $\mathfrak{g}^n = 0$.

Engel's Theorem

Lemma

If $\mathfrak{g}$ *is a subalgebra of* $\mathfrak{gl}(V)$ *consisting of only nilpotent endomorphisms, there exists some nonzero* $v \in V$ *such that* $[x, v] = 0$ *for all* $x \in \mathfrak{g}$.

Theorem (Engel)

If all elements of a Lie algbera $\mathfrak{g}$ *are ad-nilpotent, ie; there exists some* n *such that* $\mathrm{ad}^n(x) = 0$, *then* $\mathfrak{g}$ *is nilpotent.*

Root Systems

A subset $\Phi \subset E$ for Euclidian space E is a root system in E if

1. Φ if a nonzero finite subset and spans E

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2. If $\alpha \in \Phi$, the reflection σ_α leaves Φ invariant
3. If $\alpha, \beta \in \Phi$,
$$\langle \beta, \alpha \rangle \in \mathbb{Z}$$

Cartan Subalgebras

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Definition

Let $\mathfrak{c}$ be a Cartan subalgebra of a semisimple Lie algbera $\mathfrak{g}$. An element $r \in \mathfrak{c}^*$ is a root of $\mathfrak{g}$ relative to $\mathfrak{c}$ if there exists some $x \in \mathfrak{g}$ such that $[y, x] = r(y)x$ for all $y \in \mathfrak{c}$.

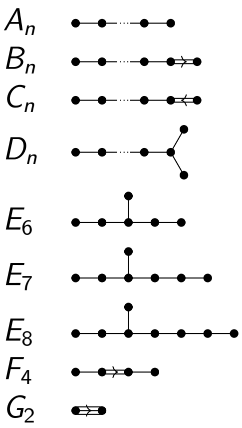
Dynkin Diagrams

Definition

The Dynkin Diagram corresponding to a root system Φ is created by first drawing a node $\circ$ for each [simple] root of Φ. The number of lines connecting two roots x and y is $2\cos(\theta)\frac{\|y\|}{\|x\|}$.

Classification Theorem

Each semisimple Lie algebra has a dynkin diagram of the following:



Thank you!