ON THE REPRESENTATIONS AND CLASSIFICATION OF SIMPLE COMPLEX LIE ALGEBRAS

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ABSTRACT. In this paper, we define Lie algebras and prove interesting theorems pertaining to their nilpotency and solvability. We then define semisimple Lie algebras and study their representations, root spaces, and root systems. We conclude with an outline for the proof of the classification theorem.

CONTENTS

1. INTRODUCTION

In the 1800s, Sophus Lie studied Lie groups to model the symmetries of equations.

Definition 1.1. A Lie group is a finite real smooth manifold that is also a group. The operations of multiplication and inversion are smooth maps.

To obtain a Lie algebra, one takes the tangent space to the manifold at its identity, e. The tangent space came with a bilinear operation which became known as the Lie bracket. It turns out that Lie algebras have many properties and associated structures analgogous to rings. For example, for a Lie algebra $\mathfrak{g}$ with the bracket operation $[.,.] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$, an ideal of $\mathfrak{g}$ is a subspace I that is closed under the bracket operation and has $[x, y] \in I$ for all $x \in I$ and $y \in \mathfrak{g}$. A simple Lie algebra is defined as an algebra with its only ideals being 0 and itself. Furthermore, a semisimple Lie algebra is defined as an algebra with having only the trivial solvable ideal. It was shown that this criterion for a semisimple Lie algebra is equivalent to the definition of a semisimple Lie algebra being the direct sum of simple Lie algebras. Wilhelm Killing and Elie Cartan studied the representation of simple Lie algebras and were able to classify the real simple Lie algebras and study their representation theory. It turned out that one could decompose a Lie algebra into "roots" in Euclidian space. From here, diagrams were created to model Lie algebras, named after the mathematician Eugene

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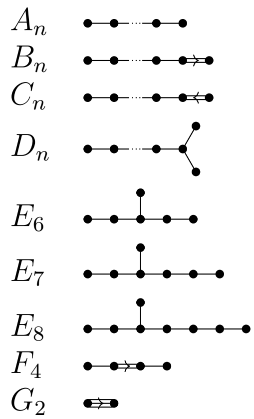
Dynkin. A beautiful theorem classified the Dynkin diagrams of simple Lie algebras into one of 9 types as shown.



The theorem as well as some of the key steps to proving it are included in this paper. Finally, we assume a working-knowledge of linear algebra at the knowledge of a first course. In the second section, we define Lie algebras and discuss various constructions associated with them. In the third section, we define solvability and nilpotency of Lie algebras and prove Engel's theorem as well as Lie's theorem. In the fourth section, we define representations of Lie algebras and examine the representations of one important simple Lie algebra. In the fifth section, we show how to decompose a Lie algebra into a root space. In the sixth section, we consider root systems in Euclidian space and describe their interconenctedness with Lie algebras. In the final section, we state and outline the proof of the classification theorem for simple Lie algebras.

2. Preliminaries

Definition 2.1. A Lie algebra $\mathfrak{g}$ over a commutative field $\mathbb{F}$ is a vector space equipped with an operation $[.,.] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ for which the following axioms hold:

(1) For all $w, x, y, z \in \mathfrak{g}$, $[w + \lambda x, y + \mu z] = [w, y + \mu z] + [\lambda x, y + \mu z] = [w, y] + \mu[w, z] + \lambda[x, y] + \lambda\mu[x, z]$ where $\lambda, \mu \in \mathbb{F}$.
(2) For all $x \in \mathfrak{g}$, $[x, x] = 0$.
(3) For all $x, y \in \mathfrak{g}$, $[x, y] = -[y, x]$.
(4) For all $x, y, z \in \mathfrak{g}$, $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$.

The bracket operation is referred to as the Lie bracket for $\mathfrak{g}$.

In other words, the Lie bracket is a bilinear map that is alternativity, anti-commutativity, and the Jacobi identity. One common misconception about Lie algebras is that their Lie bracket is associative, similar to groups. However, as this is not implied from the axioms, it is not nessecarily true that the Lie bracket is associative. We can also see that axiom (3) is implied from axiom (2) by calculating $[x+y, x+y] = [x, x]+[x, y]+[y, x]+[yy] = [x, y]+[y, x]$. By axiom (2), this must be equal to 0 which implies axiom (3). It is therefore redundant to include axiom (3) but we do so anyways to remind ourselves of this property. One may also wonder if axiom (3) implies axiom (2). Unassuming axiom (2), we have $[x, x] = -[x, x]$. This then implies that $2[x, x] = 0$ which gives $[x, x] = 0$ if $\operatorname{char} \mathbb{F} \neq 2$. From here on, all fields $\mathbb{F}$ that we mention are taken to be commutative. We also assume $\mathbb{F}$ to have characteristic zero.

Example. Let $\mathfrak{g} = \mathbb{R}^3$ and $[.,.]$ be the cross product. To show that this is a Lie algebra, we must display that each of the axioms holds for it. Axioms (1) and (2) are well known properties of the cross product. Since (2) implies (3) we do not need to show (3). For the Jacobi identity, note that if we show that it holds for the standard basis vectors of $\mathbb{R}^3$, it must hold for all other vectors in $\mathbb{R}^3$ due to bilinearity. Letting e_1, e_2, e_3 denote the standard basis vectors of $\mathbb{R}^3$, we have

$$e_1 \times (e_2 \times e_3) + e_2 \times (e_3 \times e_1) + e_3 \times (e_1 \times e_2) = 0 + 0 + 0 = 0.$$

Example. Consider $\mathfrak{g}$ to be $\mathbb{C}$ with the bracket operation representing addition, ie; $[z, w] = z + w$ for all $z, w \in \mathfrak{g}$. Then, the bracket operation is clearly not bilinear, as $[a + \lambda b, c + \mu d] = (a + \lambda b) + (c + \mu d) = [a, c] + [\lambda b, \mu d]$. Thus, $\mathfrak{g}$ is not a Lie algebra.

Example. Consider the vector space $\mathfrak{gl}(V)$, the vector space of all endomorphisms of a vector space V over $\mathbb{F}$. We define $[x, y] = xy - yx$ for all $x, y \in \mathfrak{gl}(V)$ where multiplication represents composition. As endomorphisms can be considered matrices by fixing a basis, we may also write $\mathfrak{gl}(V)$ are $\mathfrak{gl}(n, \mathbb{F})$, n by n matrices over $\mathbb{F}$, where $n = \dim(V)$. We could then take the bracket operation to be represent matrix multiplication rather than compoisiton. To show that this is a Lie algebra, axioms one and two are trivial. For the Jacobi identity, we see that

$$\begin{aligned}[x, [y, z]] + [y, [z, z]] + [z, [x, y]] &= [x, yz - zy] + [y, zx - xz] + [z, xy - yx] \\ &= x(yz - zy) - (yz - zy)x + y(zx - xz) \\ &- (zx - xz)y + z(xy - yx) - (xy - yx)z \\ &= 0\end{aligned}$$

for all $x, y, z \in \mathfrak{g}$.

Proposition 2.2. *For a Lie algebra* $\mathfrak{g}$,

$$[x, 0] = 0 = [0, x]$$

for all $x \in \mathfrak{g}$.

Proof. By the fact that $[.,.]$ is bilinear, we have

$$[x, 0] = [x, (0 \cdot x)] = 0[x, x] = 0$$

and

$$[0, x] = [(0 \cdot x), x] = 0[x, x] = 0.$$

■

Definition 2.3. A Lie subalgebra $\mathfrak{h}$ of a Lie algebra $\mathfrak{g}$ is a vector subspace of $\mathfrak{g}$ where for all $x, y \in \mathfrak{h}$, $[x, y] \in \mathfrak{h}$.

Note that a subalgebra of a Lie algebra has the same bracket structure as the original Lie algebra.

Example. Another lie algebra is the algebra $\mathfrak{sl}(2, \mathbb{C})$, the special linear algebra of dimension 2. This is defined to be all two dimensional matrices over $\mathbb{C}$ containing trace 0. To show that this is a subalgebra of $\mathfrak{gl}(2, \mathbb{C})$, we have

$$\begin{aligned}\mathrm{Tr}([x, y]) &= \mathrm{Tr}(xy - yx) \\ &= \mathrm{Tr}(xy) - \mathrm{Tr}(yx) \\ &= 0.\end{aligned}$$

This implies that for all elements in $\mathfrak{sl}(2, \mathbb{C})$, applying the inherited bracket operation from the general linear algebra returns an endomorphism with trace 0, hence an element of $\mathfrak{sl}(2, \mathbb{C})$. Let us now try to compute the dimension for this algebra. Letting e_{ij} be the 2×2 matrix with a 1 in the ith row and jth column and a 0 elsewhere, we can take the standard basis in $\mathfrak{sl}(2, \mathbb{C})$ to be

$$e_{11} - e_{22} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, e_{12} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, e_{21} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

So, the dimension of $\mathfrak{sl}(2, \mathbb{C})$ is 3. We should now consider generalizing this algebra to matrices (or endomorphisms, depending on what term you are more comfortable with), of arbitrary dimension n.

Example. The algebra $\mathfrak{sl}(n, \mathbb{F})$ is defined to be all endomorphisms from an $n-$dimensional endomorphisms of a vector space V over $\mathbb{F}$ with trace 0. We have that $\mathfrak{sl}(n, \mathbb{F})$ is a subalgebra of $\mathfrak{gl}(n, \mathbb{F})$ by the same reasoning from $\mathfrak{sl}(2, \mathbb{C})$. Taking inspiration from the algebra $\mathfrak{sl}(2, \mathbb{F})$, we should attempt to find a basis in $\mathfrak{sl}(n, \mathbb{F})$ and then the dimension. We note elements not along the main diagonal of a matrix in $\mathfrak{sl}(n, \mathbb{F})$ may be arbitary, as they do not impact the trace. So, matrices of the form e_{ij} for $1 \leq i \neq j \leq n$ are part of the standard basis of $\mathfrak{sl}(2, \mathbb{F})$. Now, the matrices with nonzero diagnoal entries and all other elements being 0 can be spanned by $e_{ij} - ei + 1, j + 1$ for $1 \leq i \leq n - 1$. Thus, combining these two sets, we have the standard basis of $\mathfrak{sl}(n, \mathbb{F})$ as

$$B = \begin{cases} e_{ij} & 1 \leq i \neq j \leq n \\ e_{ij} - e_{i+1,j+1} & 1 \leq i \leq n-1 \end{cases}$$

The dimension of $\mathfrak{sl}(n, \mathbb{F})$ is $n(n-1) + (n-1) = n^2 - 1$.

One important theorem, the proof of which is out of the scope of the paper, states that each Lie algebra $\mathfrak{g}$ can be thought of as matrices over $\mathbb{F}$ using the bracket $[x, y] = xy - yx$ for all $x, y \in \mathfrak{g}$. More precisely, the theorem states:

Theorem 2.4 (Ado)**.** *Every finite Lie algebra is isomorphic to a finite general linear algebra.*

Although we are unable to prove this theorem, an interested reader may find it in [Swa].

Definition 2.5. If $\mathfrak{g}$ and $\mathfrak{h}$ are Lie algebras, their direct sum $\mathfrak{g} \oplus \mathfrak{h}$ is the vector space direct sum with Lie bracket restricting to $[.,.]_{\mathfrak{g}}$ and $[.,.]_{\mathfrak{h}}$ with $[\mathfrak{g}, \mathfrak{h}] = 0$.

One very special subalgebra is the ideal of a Lie algebra. Its definition is analogous to the one for an ideal of a ring and thus has similar consequences.

Definition 2.6. An Ideal I of a Lie algebra $\mathfrak{g}$ is a subalgebra of $\mathfrak{g}$ where for all $x \in I$ and $y \in \mathfrak{g}, [x, y] \in I$.

Definition 2.7. A Lie algebra $\mathfrak{g}$ is simple if it does not have any ideals besides 0 and $\mathfrak{g}$. Additionally, $\mathfrak{g}$ cannot be commutative (abelian), ie; it cannot be true that $[x, y] = 0$ for all $x, y \in \mathfrak{g}$.

Example. Consider the special linear algebra of dimension $2, \mathfrak{sl}(2, \mathbb{F})$. We wish to show that this is simple. Recall the standard basis we constructed for $\mathfrak{sl}(2, \mathbb{F})$. We can denote

$$A = e_{11} - e_{22}, B = e_{12}, C = e_{21}.$$

Taking the inherited Lie bracket from $\mathfrak{gl}(2, \mathbb{F})$ we get

$$[A, B] = 2B, [A, C] = -2C, [B, C] = A.$$

Now, because $B = \{A, B, C\}$ is a basis in $\mathfrak{sl}(2, \mathbb{F})$, each element in $\mathfrak{sl}(2, \mathbb{F})$, every element in $\mathfrak{sl}(2, \mathbb{F})$ can be expressed as a linear combination of elements from B. Suppose that I is an ideal of $\mathfrak{sl}(2, \mathbb{F})$ and $x \in I$ is some element of the ideal. Then, we can write $x = c_1A+c_2B+c_3C$ for $c_1, c_2, c_3 \in \mathbb{F}$. By the definition of an ideal, we must have $[x, B] \in I$. So,

$$\begin{aligned} [x, B] &= [c_1A + c_2B + c_3C, B] \\ &= c_1[A, B] + c_2[B, B] + c_3[C, B] = c_1[A, B] + c_3[C, B] = 2c_1B - c_3A \\ &\in I. \end{aligned}$$

Bracketing this again with B gives

$$\begin{aligned} [2c_1B - c_3A, B] &= 2c_1[B, B] - c_3[A, B] \\ &= -2c_3B \\ &\in I. \end{aligned}$$

Thus, if $-2c_3 \neq 0$, B is an element of the ideal I. If $B \in I$, then we must have that $[B, C] = A$ is an element of the ideal and then $[A, C] = -2C$ imples that C must also be an ideal. Since I is an algebra of its own right, this would imply that $I = \mathfrak{sl}_2(2, \mathbb{F})$. So, we must have that $c_3 = 0$. Going back to x, we have

$$\begin{aligned} [x, C] &= [c_1A + c_2B + c_3C, C] \\ &= c_1[A, C] + c_2[B, C] + c_3[C, C] \\ &= -2c_1C + c_2A \\ &\in I. \end{aligned}$$

Then, we must also have

$$\begin{aligned} [-2c_1C + c_2A, C] &= -2c_1[C, C] + c_2[A, C] \\ &= -2c_2C \\ &\in I. \end{aligned}$$

Similar to our previous argument, in order for $I \neq \mathfrak{sl}(2, \mathbb{F})$, we must have that $c_2 = 0$. If we have $c_2 = c_3 = 0$, then $x = c_1A$. But, if c_1 is nonzero, then A is an element of I, and we can recover the other elements of the basis by bracketing A with B and C. Thus, for I to not be equal to $\mathfrak{sl}(2, \mathbb{F})$, we have $c_1 = 0$. But, if all elements of I are equal to 0, then $I = 0$. We have shown that I is either equal to 0 or $\mathfrak{sl}(2, \mathbb{F})$, proving that $\mathfrak{sl}(2, \mathbb{F})$ is simple.

Definition 2.8. A linear transformation $\phi : \mathfrak{g} \to \mathfrak{h}$ between two Lie algebras is said to be a Lie homomorphism if

$$\phi([x, y]) = [\phi(x), \phi(y)]$$

for all $x, y \in \mathfrak{g}$.

A homomorphism that is both injective and surjective is an isomorphism. Lie Isomorphisms are fairly powerful as the Ring isomorphism theorems may be applied to Lie isomorphisms.

Definition 2.9. The adjoint of an element x in a Lie algebra $\mathfrak{g}$ is the map $\operatorname{ad} x : \mathfrak{g} \to \mathfrak{g}$ defined by $adx(y) = [x, y]$ for all $y \in \mathfrak{g}$.

Remark 2.10. The adjoint $\operatorname{ad} x$ is not a bilinear map but a linear map.

Definition 2.11. The adjoint representation of a Lie algebra is the map that sends each element of a Lie algebra to its adjoint. In other words, the adjoint representation is the map $\operatorname{ad} : \mathfrak{g} \to \mathfrak{gl}(V)$ defined by $\operatorname{ad}(x) = \operatorname{ad} x$ for all $x \in \mathfrak{g}$. The image of ad is denoted as $\operatorname{ad}(\mathfrak{g})$.

To show that this is a Lie homomorphism, we have for all $x, y, z \in \mathfrak{g}$,

$$\begin{aligned}[\operatorname{ad} x, \operatorname{ad} y](z) &= (\operatorname{ad} x \operatorname{ad} y - \operatorname{ad} y \operatorname{ad} x)(z) \\ &= [x,[y,z]] - [y,[x,z]] = [x,[y,z]] \qquad \text{Jacobi Identity} \\ &= \operatorname{ad}([x,y])z.\end{aligned}$$

Somtimes, when dealing with multiple algebras and subalgebras, it may be ambiguous to which space we are referring ad to act on. To solve this dilemma, we may use subscript notation $\operatorname{ad}_{\mathfrak{g}}$ to denote the space ad is acting on. However we only use this notation when there is some ambiguity.

Definition 2.12. The derived algebra of a Lie algebra $\mathfrak{g}$ is the set $[\mathfrak{g}, \mathfrak{g}]$ consisting of all linear combinations of lie brackets $[x, y]$ for all $x, y \in \mathfrak{g}$.

Example. Consider $\mathfrak{g} = \mathfrak{sl}(n, \mathbb{F})$. Recalling the formula for the bracket of two elements of $\mathfrak{sl}(n, \mathbb{F})$, we notice that we can recreate each element of its standard basis. Thus, we have $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$.

Definition 2.13. If $[\mathfrak{g}, \mathfrak{g}] = 0$, a Lie algebra $\mathfrak{g}$ is said to be simple.

Definition 2.14. The normalizer of a subalgebra $\mathfrak{h}$ of a Lie algebra $\mathfrak{g}$, is the set $N_{\mathfrak{g}}(\mathfrak{h}) = \{x \in \mathfrak{g} | [x, y] \in \mathfrak{h}, \forall y \in \mathfrak{h}\}$

Restated, the normalizer of a subalgebra is simply the largest subalgebra of $\mathfrak{g}$ containing $\mathfrak{h}$ as a basis.

Definition 2.15. For a Lie algebra $\mathfrak{g}$ and one of its ideals, I, the quotient algebra $\mathfrak{g}/I$ is defined to be the quoteint space $\mathfrak{g}/I$ with Lie bracket $[x + I, y + I] = [x, y] + I$.

One might wonder if it is true that

$$[x + I, y + I] \neq [x' + I, y + I]$$

when x and x' are related under I. If this were to be true, the definition of the bracket of a quotient algebra would be not well defined. To show that is well defined suppose that x' and y' are related to two elements of a Lie algebra $\mathfrak{g}, x$ and y respectively. By definition of relation, we have $x' = x + v$ and $y' = y + u$ for $v, u \in I$. We then have

$$\begin{aligned}[x' + I, y' + I] &= [(x + v) + I, (y + u) + I] \\ &= [x + v, y + u] + I \\ &= [x, y] + [x, u] + [v, y] + [v, u] \\ &= [x, y] + 0 + I \\ &= [x, y] + I \\ &= [x + I, y + I].\end{aligned}$$

Definition 2.16. For a Lie algebra $\mathfrak{g}$ and one of its ideals I, the canonical map $\pi : \mathfrak{g} \to \mathfrak{g}/I$ is defined to be $\pi(x) = x + I = [x]$ for all $x \in \mathfrak{g}$.

From this definition, we can consider the standard isomorphism theorems for rings. We may easily replicate the proofs to show that some of the theorems hold true for Lie algebras as well. For example, we have that if I and J are ideals of a Lie algebra $\mathfrak{g}$, then $\frac{I+J}{J} \cong \frac{I}{I\cap J}$. Additionally, if $I \subset J$ are ideals of a Lie algebra $\mathfrak{g}$, then J/I is an ideal of L/I and $\frac{\frac{L}{I}}{\frac{J}{I}} \cong \frac{L}{J}$.

3. Solvability and Nilpotency

Definition 3.1. The derived series of a Lie algebra $\mathfrak{g}$ is the series

$$L^{(0)} = L, L^{(1)} = [L^{(0)}, L^{(0)}], \dots, L^{(i-1)} = [L^{(i-2)}, L^{(i-2)}, L^{(i)} = [L^{(i-1)}, L^{(i-1)}].$$

Definition 3.2. A Lie algebra $\mathfrak{g}$ is solvable if there exists some i such that $L^{(i)} = 0$.

Lemma 3.3. *Suppose that $\mathfrak{g}$ is a solvable Lie algebra. Then, all subalgebras of $\mathfrak{g}$ are solvable. Additionally, if $\mathfrak{h}$ is a Lie algebra for which there exists a Lie homomorphism $\phi : \mathfrak{g} \to \mathfrak{h}$, then $\mathfrak{h}$ is solvable.*

Proof. Let $\mathfrak{h}$ be a subalgebra of $\mathfrak{g}$. Then, if $x, y \in \mathfrak{h}, x, y \in \mathfrak{g}$, so we have $\mathfrak{h}^{(i)} \subset \mathfrak{g}^{(i)}$. So if $\mathfrak{g}^{(n)} = 0$ we also have $\mathfrak{g}^{(n)} = 0$ which completes the first part of the lemma. For the second part, suppose that $\mathfrak{h}$ is a homomorphic image of $\mathfrak{g}$ with respect to the surjective Lie homomorphism $\phi : \mathfrak{g} \to \mathfrak{h}$. We claim that $\phi(\mathfrak{g}^{(i)}) = \mathfrak{h}^{(i)}$. We show this with induction on i. For $i = 0$, we have

$$\phi(\mathfrak{g}^{(0)}) = \phi(\mathfrak{g}) = \mathfrak{h}$$

as ϕ is surjective. If we assume that the statement holds for i, we have

$$\phi(\mathfrak{g}^{(i+1)}) = \phi([\mathfrak{g}^{(i)}, \mathfrak{g}^{(i)}]) = [\phi(\mathfrak{g}^{(i)}), \phi(\mathfrak{g}^{(i)})] = [\mathfrak{h}^{(i)}, \mathfrak{h}^{(i)}] = \mathfrak{h}^{(i+1)}.$$

■

Proposition 3.4. *If $\mathfrak{g}$ is not necessarily a Lie algebra, and I is a solvable ideal of $\mathfrak{g}$ such that the quotient algebra $\mathfrak{g}/I$ is solvable, then $\mathfrak{g}$ is solvable.*

Proof. Recall the canonical homomorphism $pi : \mathfrak{g} \to \mathfrak{g}/I$. Taking n to be the number such that $(\mathfrak{g}/I)^{(n)} = 0$, we have $\pi(\mathfrak{g}^{(n)}) = (\mathfrak{g}/I)^{(n)} = 0$ by the proof of the prior lemma. Thus by the definition of a quotient space we have $\mathfrak{g}^{(n)} \subset I$. Taking m to be the value such that $I^{(m)} = 0$, we now have $\mathfrak{g}^{(n)^{(m)}} \subset I^{(m)} = 0$, implying the result. ■

Corollary 3.5. *Suppose that $\mathfrak{g}$ is a Lie algebra containing solvable ideals I and J. Then, $I + J$ is a solvable ideal of $\mathfrak{g}$.*

Proof. Recall that we have $\frac{I+J}{J} \cong \frac{I}{I \cong J}$. By the lemma, we have that $(I + J)/J$ is solvable. Since J is a solvable ideal of $I + J$ such that $(I + J)/J$ is solvable, we have that $I + J$ is solvable directly from the proposition. ■

When defining the derived series, we chose that for $\mathfrak{g}^{(i)}$ we would bracket $\mathfrak{g}^{(i-1)}$ with $\mathfrak{g}^{(i-1)}$. We may wonder what would happen if we chose the second argument of the bracket to always be $\mathfrak{g}$ rather than some subalgbera of $\mathfrak{g}$.

Definition 3.6. The descending central series of a Lie algebra $\mathfrak{g}$ is the series

$$\mathfrak{g}^0 = \mathfrak{g}, \dots, \mathfrak{g}^i = [\mathfrak{g}, \mathfrak{g}^{i-1}].$$

A lie algebra $\mathfrak{g}$ is nilpotent if there exists some n such that $\mathfrak{g}^n = 0$. Alternatively, we could have expressed this definition as for $\mathfrak{g}$ to be nilpotent, we must have that

$$[\mathfrak{g}, \dots, [\mathfrak{g}, \mathfrak{g}], \dots] = 0$$

for bracking $\mathfrak{g}$ with itself some number of times. So if $\mathfrak{g}$ is nilpotent, then for all sequences of elements in $\mathfrak{g}$, such as $\{x_1, \dots, x_n\}$, we have

$$[x_n, [x_{n-1}, \dots, [x_1, y], \dots] = 0$$

for all $y \in \mathfrak{g}$. Letting $x_n = x_{n-1} = \cdots = x_1 = x$ for some $x \in \mathfrak{g}$, we can see that if $\mathfrak{g}$ is nilpotent, then $(\operatorname{ad} x)^n = 0$, ie; $\operatorname{ad} x$ is nilpotent. We say that x is ad nilpotent if $\operatorname{ad} x$ is nilpotent. From this reasoning, if $\mathfrak{g}$ is nilpotent, all elements $x \in \mathfrak{g}$ are ad nilpotent. We may now wonder if the converse is also true, if all elements of a Lie algebra $x \in \mathfrak{g}$ are ad nilpotent, then is $\mathfrak{g}$ nilpotent? This is known as Engel's Theorem, which we will prove soon.

We present some statements which are analogous to the statements made for solvable Lie algebras.

Proposition 3.7. *If $\mathfrak{g}$ is a nilpotent Lie algebra, then all subalgebras of $\mathfrak{g}$ are nilpotent. Additionally, all Lie algebras $\mathfrak{h}$ for which there exists a Lie homomorphism $\phi : \mathfrak{g} \to \mathfrak{h}$ are nilpotent.*

Proof. Letting $\mathfrak{h}$ be a subalgebra of $\mathfrak{g}$, we have $\mathfrak{h}^i \subset \mathfrak{g}^i$. Thus we have proved the first part of the proposition. For the second part, we consider the surjective homomorphism $\phi : \mathfrak{g} \to \mathfrak{h}$ where $\mathfrak{h}$ is some Lie algebra. Then, we claim that $\phi(\mathfrak{g}^i) = \mathfrak{h}^i$. We prove this using idnuction on i. For $i = 0$, we have $\phi(\mathfrak{g}^0) = \phi(\mathfrak{g}) = \mathfrak{h}$ as ϕ is surjective. Assuming the homomorphism for i, we have

$$\phi(\mathfrak{g}^{i+1}) = \phi([\mathfrak{g}, \mathfrak{g}^i]) = [\phi(\mathfrak{g}), \phi(\mathfrak{g}^i)] = [\mathfrak{h}, \mathfrak{h}^i] = \mathfrak{h}^{i+1}.$$

■

Proposition 3.8. *Suppose that $\mathfrak{g}$ is a Lie algebra. Define the center of a Lie algebra $\mathfrak{g}$ to be*

$$C(\mathfrak{g}) = \{x \in \mathfrak{g} | [x, z] = 0, \forall z \in \mathfrak{g}\}.$$

Then, if the quotient algebra $\mathfrak{g}/Z(\mathfrak{g})$ is nilpotent, $\mathfrak{g}$ is nilpotent.

Proof. Because$\mathfrak{g}/Z(\mathfrak{g})$ is nilpotent, we have $(\mathfrak{g}/Z(\mathfrak{g}))^n = 0 + Z$ for some n. Note that this 0 is an element of $\mathfrak{g}$ rather than $Z(\mathfrak{g})$. Taking the canonical homormorphism, we have $\pi(\mathfrak{g}^n) = (\mathfrak{g}/Z(\mathfrak{g}))^n = 0 + Z(\mathfrak{g})$. So, we have $\mathfrak{g}^n \subseteq \ker(\pi) = Z(\mathfrak{g})$ by the definition of the center. To complete the proof, we have $\mathfrak{g}^{n+1} = [\mathfrak{g}, \mathfrak{g}^n] \subseteq [L, Z] = 0$ so $\mathfrak{g}$ is nilpotent. ■

Proposition 3.9. *If a lie algebra $\mathfrak{g}$ is nilpotent, its center $Z(\mathfrak{g})$ is not zero.*

Proof. There exists some n such that $\mathfrak{g}^n = 0$ by the deifnition of nilpotency. Because if $\mathfrak{g}^n = 0, \mathfrak{g}^{n+1} = 0$, we take the smallest such n that $\mathfrak{g}^n = 0$. So, we have $\mathfrak{g}^n = [\mathfrak{g}, \mathfrak{g}^{n-1}] = 0$, each element of $\mathfrak{g}^{n-1}$ bracketed with each element of $\mathfrak{g}$ returns 0, implying that $\mathfrak{g}^{n-1} \subseteq Z(\mathfrak{g})$. But, because we previously said that n is the smallest number such that $\mathfrak{g}^n = 0$, we have that $\mathfrak{g}^{n-1} \neq 0$, completing the proof. ■

In order to prove Engel's theorem, we must first prove the following lemmas.

Lemma 3.10. *Let $x \in \mathfrak{gl}(V)$ for finite V be nilpotent. Then, there exists some $v \in V$ such that $xv = 0$.*

Proof. Take n to be the smallest number such that $x^n = 0$. Then, $x^{n-1} \neq 0$. So, we have that $x^{n-1}V \neq \{0\}$ where $x^{n-1}V$ denotes the set of multiplying x^{n-1} with each element of V. Since $x^{n-1}V \neq \{0\}$, there must exist some $w \in V$ such that $x^{n-1} = w \neq 0$. Let's call this quantity u. So to complete the proof, we use left multiplication by x to obtain

$$xu = x(x^{n-1}v) = x^n v = 0.$$

Thus u is the desired element in V such that $xu = 0$. ∎

Lemma 3.11. *Let $x \in \mathfrak{gl}(V)$ for finite V be nilpotent. Then,* $\operatorname{ad} x$ *is nilpotent.*

Proof. Take n to be the smallest n such that $x^n = 0$. Then, because we are working over the general linear algebra, we have $\operatorname{ad} x(y) = xy - yx$ for all $y \in \mathfrak{g}$. We can now denote left multiplication by x as L_x and right multiplication of x by R_x. For any element $y \in \mathfrak{g}$, we have $L_x R_x y = xyx = R_x L_x y$ as matrix multiplication is associative. Thus, L_x and R_x commute. Using the binomial theorem, we have

$$\begin{aligned}(\operatorname{ad} x)^{2n}(y) &= (L_x - R_x)^{2n}(y) \\ &= \sum_{i=0}^{2n} \binom{2n}{i} (-1)^{2n-i} L_x^i R_x^{2n-i} y \\ &= \sum_{i=0}^{2n} \binom{2n}{i} (-1)^{2n-i} x^i y x^{2n-i} \\ &= 0.\end{aligned}$$

We computed the last expression by noting that since i and $(2n - i)$ sum to $2n$, one of them has to be greater than or equal to n. Since $x^n = 0$, the last expression must be equal to 0, showing that $\operatorname{ad} x$ is nilpotent. ∎

Lemma 3.12. *If $\mathfrak{g}$ is a linear Lie algebra consisting of nilpotent endomorphisms, there exists some nonzero $v \in V$ such that $xv = 0$ for all $x \in \mathfrak{g}$.*

Proof. We proceed with strong induction on $n = \dim \mathfrak{g}$. The case $n = 0$ trivial and the case $n = 1$ is implied from Lemma 3.1. Now let $\mathfrak{h}$ be a maximal proper subalgebra of $\mathfrak{g}$, a proper subalgebra that is not contained in any other subalgbera of $\mathfrak{g}$. We wish to decompose $\mathfrak{g}$ into the direct sum of $\mathfrak{h}$ and some other algebra. By the lemma we know that for each $x \in \mathfrak{h}, \operatorname{ad} x$ is nilpotent. So, we consider the restricted adjoint representation on $\mathfrak{g}$ defined by $\operatorname{ad}_{\mathfrak{h}} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ where the domain is restricted to elements of $\mathfrak{h}$. So, we may take the restricted adjoint representation $\operatorname{ad}_{\mathfrak{h}} : \mathfrak{g}/\mathfrak{h} \to \mathfrak{gl}(\mathfrak{g}/\mathfrak{h})$. As we have $\dim(\mathfrak{h}) < \dim(\mathfrak{g})$, we can use the induction hypothesis to say that there is some $x + \mathfrak{h} \in \mathfrak{g}/\mathfrak{h}$ such that $[y + \mathfrak{h}, x + \mathfrak{h}] = 0$ for all $y \in \mathfrak{h}$. We can also express this as $[y + \mathfrak{h}, x + \mathfrak{h}] = [y, x] + \mathfrak{h} = 0$ which implies that $[y, x] \in \mathfrak{h}$ for all $y \in \mathfrak{h}$. The preceding argument shows that $N_{\mathfrak{h}}(\mathfrak{g})$ contains $\mathfrak{h}$ and since $\mathfrak{h}$ is maximal, we have $N_{\mathfrak{h}}(\mathfrak{g}) = \mathfrak{g}$. Hence, $\mathfrak{h}$ is an ideal of $\mathfrak{g}$. We have $\dim(\mathfrak{g}/\mathfrak{h}) = 1$ so $\mathfrak{h}$ has dimension 1 which allows us to write $\mathfrak{g} = \mathfrak{h} \oplus \mathbb{F}z$ for $z \in \mathfrak{g} - \mathfrak{h}$. As $\mathfrak{h}$ has a smaller dimension than $\mathfrak{g}$, we have $W = \{v \in V | \mathfrak{h}v = 0\}$ having a nonzero cardinality. Because $\mathfrak{h}$ is an ideal, we have $yx(w) = xy(w) - [x, y](w) = 0$. Referring back to our decomposition of $\mathfrak{g}$, let z be a endomorphism with an eigenvector $w \in W$ for z acting on W. This w satisfies $x(w) = 0$ for all $x \in \mathfrak{g}$, completing the proof. ∎

Theorem 3.13 (Engel)**.** *If all elements of a Lie algbera $\mathfrak{g}$ are ad-nilpotent, then $\mathfrak{g}$ is nilpotent.*

Proof. By the prior lemma, there exists some $v \in \mathfrak{g}$ such that $[x, v] = 0$ for all $v \in \mathfrak{g}$. Thus, the center of the Lie algebra, $Z(\mathfrak{g})$, is nonzero. Inducting on $\dim \mathfrak{g}$ proves that the quotient algebra $\mathfrak{g}/Z(\mathfrak{g})$ is nilpotent so $\mathfrak{g}$ is nilpotent by Proposition 3.8. ■

Corollary 3.14. *If $\mathfrak{h}$ is a nonzero ideal of a nilpotent Lie algebra $\mathfrak{g}$, then $\mathfrak{h} \cap Z(\mathfrak{g})$ is nonzero.*

Proof. Because $\mathfrak{g}$ acts on $\mathfrak{h}$ by ad, Engel's theorem guarantees some $x \in \mathfrak{h}$ such that $[y, x] = 0$ fr all $y \in \mathfrak{h}$. The corollary then follows from the definition of the center of a Lie algebra. ■

A flag is a sequence of subsets $0 = V_0 \subset V_1 \subset \cdots \subset V_n = V$ where V_i has dimension i. A endomorphism $x \in \mathfrak{gl}(V)$ stabilizes this flag if $x(V_i) \subset V_i$ for all V_i in the sequence.

Proposition 3.15. *Taking the same conditions as Engel's theorem, there exists a flag (V_i) in V that is stabilized by $\mathfrak{g}$ such that $x(V_i) \subset V_{i-1}$ for all $x \in \mathfrak{g}$.*

By Engel's theorem, there is some $v \in V$ such that $x(v) = 0$ for all $x \in \mathfrak{g}$. Let $V_1 = \mathbb{F}v$. We can now let $W = V/V_1$ and thus the action of $\mathfrak{g}$ on W is by nilpotent endomorphisms. To show that W has a flag stabilized by $\mathfrak{g}$, we use induction on $\dim V$. The inverse image in V is the flag satisfying the conditions of the theorem.

Moving on from solvability, we discuss nilpotency by the means of Lie's theorem. As with Engel's theorem, we must break the theorem into shorter lemmas as it is highly technical.

Lemma 3.16. *If $\mathfrak{g}$ is a solvable Lie subalgebra of $\mathfrak{gl}(V)$ where V is a finite, nonzero vector space, there is some $v \in V$ such that v is an eigenvector for each endomorphism in $\mathfrak{g}$.*

Proof. Once again, we use strong induction on $\dim(\mathfrak{g})$. We take $n = 0$ and $n = 1$ as our base cases. As these cases are trivial, we may now show the general case. Since $\mathfrak{g}$ is solvable, we have that $[\mathfrak{g}, \mathfrak{g}]$ is included in $\mathfrak{g}$. Now we must show that $\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}]$ is abelian. We have $[x + [\mathfrak{g}, \mathfrak{g}], y + [\mathfrak{g}, \mathfrak{g}]] = [x, y] + [\mathfrak{g}, \mathfrak{g}]$ for all $x, y \in \mathfrak{g}$. Since $[x, y] \in [\mathfrak{g}, \mathfrak{g}]$, we have $[x + [\mathfrak{g}, \mathfrak{g}], y + [\mathfrak{g}, \mathfrak{g}]] = 0 + [\mathfrak{g}, \mathfrak{g}] = [0]$ which immediately implies commutativity. Thus, any subspace of the quotient $\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}]$ is an ideal. We can now construct an ideal $\mathfrak{h}$ in $\mathfrak{g}$ having codimension one by taking the inverse under the canonical homomorphism from a subspace of codimension one in $\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}]$. Since $\mathfrak{h}$ has a smaller dimension than $\mathfrak{g}$, the induction hypothesis guarantees some common eigenvector $v \in V$ for $\mathfrak{h}$. If $\mathfrak{h}$ is the zero space, we simply take v to be an eigenvector of a basis vector of $\mathfrak{g}$ to complete the proof. However, if $\mathfrak{h}$ is nonzero, we have $x(v) = \lambda(x)v$ where $\lambda : \mathfrak{h} \to \mathbb{F}$ is linear. We can construct the space $W = \{v \in V | x(v) = \lambda(x)v\}$ for all $x \in \mathfrak{h}$. By the induction hypothesis, as well as our assumption of the non-triviality of $\mathfrak{h}$, we have that W is nonzero. We must now show that W is invariant under V. We have $yx(v) = xy(v) - [x, y](v) = \lambda(y)x(v) - \lambda([x, y])(v)$ for all $x \in \mathfrak{g}, y \in \mathfrak{h}$, and $v \in W$. In order for W to be invariant under W we should have $yx(v) = \lambda(y)x(v)$. Thus, we need to show that $\lambda([x, y])(v) = 0$. For $x \in \mathfrak{g}, v \in W$, let c be the smallest positive integer such that the vectors $v, x(v), \dots, x^n(v)$ are linearly dependent. And then we take $W_i = \operatorname{span}(v, x(v), \dots, x^{i-1}(v))$. By convention we let $W_0 = 0$. Thus, we have $\dim(W_n) = n$. By our definition of n, we have that W_{n+1} is linearly dependent, so we can conclude that $W_{n+1} = W_n$. Additionally, for all $z \in W_n$, we have $x(z) \in W_n$. Now take some $y \in \mathfrak{h}$. For W_n we fix the standard basis $v, x(v), \dots, x^{n-1}(v)$ and write y as a matrix with respect to this basis. We claim that y is an upper triangular matrix with all diagonal entires equal to $\lambda(y)$. To prove, this we must first prove the claim that $yx^i(v) \equiv \lambda(y)w^i(v)$

(mod W_i). We prove this with induction on i. For $i = 0$, we have $W_i = 0$ so the claim follows. Now, we assume that the claim holds for $i-1$. We have $yx^i(v) = yxx^{i-1}(v) = xyx^{i-1}(v)$. From our inductive hypothesis, we have $yx^{i-1}(w) = \lambda(y)x^{i-1}(v) + d$ for some $d \in W_{i-1}$. We complete our proof of this claim by noticing that x sends W_{i-1} into W_i. If we are taking y to be an endomorphism acting on W_n we have an n by n matrix so $\mathrm{Tr}(y) = n\lambda(y)$. Taking some $u \in \mathfrak{h}$, we similarly have that u as well as y leaves W_n invariant so we take $[x, y]$ to act on W_n. We compute $\mathrm{Tr}([x, y]) = \mathrm{Tr}(xy - yx) = \mathrm{Tr}(xy) - \mathrm{Tr}(yx) = 0$. We computed this bracket by recalling that we are working over a subalgbera of a general linear algebra. As $[x, y]$ has a trace of 0, we have $n\lambda([x, y]) = 0$ since we are assuming $\mathbb{F}$ has characteristic 0 which means it cannot have characteristic n (refer back to our definition of n, we defined n to be positive). We have proven our prior claim that W is invariant under V. So, as in our proof of Engel's theorem, we decompose $\mathfrak{g}$ as $\mathfrak{g} = \mathfrak{h} \oplus \mathbb{F}z$ for $z \in \mathfrak{g} - \mathfrak{h}$. We take $v \in W$ to be an eigenvector of z. We see that v is an eigenvector for all elements of $\mathfrak{g}$ concluding the proof. ∎

Theorem 3.17 (Lie)**.** *If $\mathfrak{g}$ is a solvable Lie subalgebra of $\mathfrak{gl}(V)$ for finite V, $\mathfrak{g}$ stabilizes some flag in V.*

Proof. We obtain this by inducting on $\dim V$. ∎

Corollary 3.18. *If $\mathfrak{g}$ is a solvable Lie algebra, then $x \in [L, L]$ implies that $\mathrm{ad}_L\, x$ is nilpotent.*

Recall the Jordan-Chevalley decomposition of a linear operator x as $x = x_s + x_n$ where x_s is semisimple and x_n is nilpotent with both parts commuting. The decomposition theorem guarantees the uniqueness and existence of these parts. Recall that a linear operator is semisimple if its roots of its minimal polynomial over $\mathbb{F}$ are distinct. Additionally, we can express the semisimple and nilpotent parts as $x_s = p(T), x_n = q(T)$, where p and q are in one indeterminate and have no constant term. So if some endomorphism y commutes with x, so do x_s and x_n. For the final part of the decomposition, if $A \subset B \subset V$ and $x \in \mathfrak{gl}(V)$ with x mapping B into A, then x_s and x_n map B into A.

Lemma 3.19. *Let x be an endomorphism of a finite vector space V. Letting its Jordan decomposition be $x = x_s + x_n$, then $\mathrm{ad}\, x = \mathrm{ad}\, x_s + \mathrm{ad}\, x_n$ is the Jordan decomposition of $\mathrm{ad}\, x$.*

Proof. We have $[\mathrm{ad}\, x_s, \mathrm{ad}\, x_n] = \mathrm{ad}[x_s, x_n] = 0$ so $\mathrm{ad}\, x_s$ and $\mathrm{ad}\, x_n$ commute. By Lemma, if x_n is nilpotent, then $\mathrm{ad}\, x_n$ is also nilpotent. We must now show that $\mathrm{ad}\, x_s$ is semisimple. To prove this, we let $(v_1, \ldots, v_n)$ be a basis in V such that x has diagonal $(a_1, \ldots, a_n)$ where the a_is are the eigenvalues of x. Now we must take a standard basis in $\mathfrak{gl}(V)$ which we do by taking $\{e_{ij}\}$ defined by $e_{ij}(v_k) = \delta_{jk}(v_k)$. Now, because $[e_{ij}, e_{kl}] = \delta_{jk}e_{il} - \delta_{li}e_{kj}$, we have $\mathrm{ad}\, x(e_ij) = (a_1 - a_j)e_{ij}$ implying that $\mathrm{ad}\, x$ is diagonalizable and hence semisimple. ∎

Lemma 3.20. *Let W and V be vector spaces such that $W \subset V \subset \mathfrak{gl}(V)$ and W and V are Lie subalgebras. Let $S = \{x \in \mathfrak{gl}(V) | [x, b] \in A\}$ for all $b \in B$. Then, if $x \in S$ satisfies $\mathrm{Tr}(xy) = 0$ for all $y \in S, x$ is nilpotent.*

The proof of this lemma can be found in [HUM94]. Now, for one of the highlights of this paper, we prove Cartan's criterion for a solvable linear Lie algbera. Cartan's theorem tells us when a linear algebra is solvable.

Theorem 3.21 (Cartan)**.** *Suppose that $\mathfrak{g}$ is a subalgebra of $\mathfrak{gl}(V)$ for finite V. If $\mathrm{Tr}(x, y) = 0$ for all $x \in [\mathfrak{g}, \mathfrak{g}]$ and $y \in \mathfrak{g}$, then $\mathfrak{g}$ is solvable.*

Proof. First note that by Engel's theorem, if for each $x \in [\mathfrak{g}, \mathfrak{g}]$ $\operatorname{ad} x$ is nilpotent, then $[\mathfrak{g}, \mathfrak{g}]$ is also nilpotent. Furthermore, if $[\mathfrak{g}, \mathfrak{g}]$ is nilpotent, then $\mathfrak{g}$ is solvable. THus we need to prove that $[\mathfrak{g}, \mathfrak{g}]$ is nilpotent. To use the previous lemma, we take $A = [\mathfrak{g}, \mathfrak{g}] \subset B = \mathfrak{g} \subset \mathfrak{gl}(V)$. Thus, referring to the lemma, we have $M = \{x \in \mathfrak{gl}(V) | [x, \mathfrak{g}] \subset [\mathfrak{g}, \mathfrak{g}]\}$. We see that we have $\operatorname{Tr}(xy) = 0$ for all $x \in [\mathfrak{g}, \mathfrak{g}], y \in M$. Letting, $[x, y]$ be an element of a spanning set of $[\mathfrak{g}, \mathfrak{g}]$, and taking some $z \in M$, using the Lie bracket for linear algebras and standard properties of the trace function gives $\operatorname{Tr}([x, y]z) = \operatorname{Tr}(x[y, z]) = \operatorname{Tr}([y, z]x)$. And since $[y, z] \in [\mathfrak{g}, \mathfrak{g}]$ we have $\operatorname{Tr}([y, z]x) = 0$, completing the proof. ■

4. Representation Theory

We may turn our attention to study the representations of semisimple Lie algebras to learn more about their structures. Informally, a representation of a Lie algebra is a homomorphism sending elements of the algebra to a map that acts on the algebra. Thus, a representation allows us to have an element of a Lie algebra act on the algebra. One example of a representation is the adjoint representation, $\operatorname{ad} : \mathfrak{g} \to \mathfrak{gl}(V)$. One may also consider other representations of a Lie algebra.

Definition 4.1. A representation of a Lie algebra $\mathfrak{g}$ is a Lie homomorphism $\phi : \mathfrak{g} \to \mathfrak{gl}(V)$.

One goal of representation theory is to distinguish between representations. We hope to categorize representations. To do this, we need to include some additional theory.

Definition 4.2. If $\phi : \mathfrak{g} \to \mathfrak{gl}(V)$ is a representation of a Lie algebra $\mathfrak{g}$, $W \subset V$ is invariant under V if it is preserved under the action of $\mathfrak{g}$.

A sub-representation of a representation ϕ is a representation restricted to some invariant subspace of V. We can now distinguish between reducible and irreducible representations. We say a representation is irreducible if it has no proper nontrivial sub-representations, or in other words, the only invariant subspaces of V are the zero space and itself. On the contrary, a representation is completely reducible if there exist irreducible invariant subspaces of V such that $V = \oplus_i U_i$. We define the representation of a direct sum to be the direct sum of the representations: $x \in \mathfrak{g}$ acts on $V = \oplus_i U_i$ as $x(u_1, \ldots, u_i) = (x(u_1), \ldots, x(u_i))$. One celebrated theorem pertaining to the theory of irreducible and reducible representations is Weyl's theorem on complete reducibility, the proof of which can be found in [HUM94].

Theorem 4.3 (Weyl)**.** *If ϕ is a representation of a semisimple Lie algebra, then ϕ is completely reducible.*

This theorem simplifies the representation theory of semisimple Lie algebras and gives us some motivation to work with them. We define a special bilinear form to work with semisimple Lie algebras, the killing form $K(.,.)$.

Definition 4.4. In the future, we may use $(.,.)$ to denote the Killing form rather than $K(.,.)$. For a Lie algebra $\mathfrak{g}$, its killing form is $K(x, y) = \operatorname{Tr}(\operatorname{ad} x \operatorname{ad} y)$.

This form is bilinear, symmetric, and associative with respect to the bracket on $\mathfrak{g}$, as in $K([x, y], z) = K(x, [y, z])$.

Definition 4.5. If B is a bilinear form on a space V, its radical is

$$\operatorname{rad} B = \{v \in V | B(v, u) = 0, \forall u \in V\}.$$

A form is nondegenerate if $\operatorname{rad} B = 0$.

The radical of a bilinear form is somewhat analogous to the center of a Lie algebra. Recall that the only solvable ideals of semisimple Lie algebras are trivial.

Theorem 4.6 (Cartan's Solvability Criterion)**.** *A Lie algebra $\mathfrak{g}$ is solvable iff its Killing form, K, satisfies $K(x, y) = 0$ for all $x \in \mathfrak{g}, y \in [\mathfrak{g}, \mathfrak{g}]$.*

This theorem is interesting as it relates a Lie algebra's Killing form to the solvability of the algbera. This theorem also allows us to make a statement for the radical of the Killing form.

Proposition 4.7. *If $\mathfrak{g}$ is a Lie algebra with Killing form K, $\operatorname{rad} K$ is a solvable ideal of $\mathfrak{g}$.*

Proof. Suppose that $x \in \operatorname{rad} K$ and $y, z \in \mathfrak{g}$. Then,

$$K([x, y], z) = K(x, [y, z]) = 0$$

which tells us that $[y, z] \in \operatorname{rad} K$ by the definition. So, $\operatorname{rad} K$ is an ideal of $\mathfrak{g}$. To show that it is solvable, we consider the Killing form on the radical of the Killing form of $\mathfrak{g}$ which is certainly a mouthful. Let L denote the Killing form on $\operatorname{rad} K$. Let us decompose $\mathfrak{g}$ as $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{s}$ for Lie algebra $\mathfrak{h}$ and vector space $\mathfrak{s}$. Consider some $x \in \operatorname{rad} K$. Then, the matrix of $\operatorname{ad} x$ with respect to the Lie algbera decomposition mentioned earlier has a bottom row of zeroes. So, for all $x, y \in \operatorname{rad} K$, $L(x, y) = \operatorname{Tr}(\operatorname{ad} x \operatorname{ad} y|_{\operatorname{rad} K}) = \operatorname{Tr}(\operatorname{ad} x \operatorname{ad} y) = K(x, y) = 0$. The result follows from Cartan's Solvability Criterion. ■

Theorem 4.8 (Cartan's Semisimplicity Criterion)**.** *If $\mathfrak{g}$ is a Lie algbera, it is semisimple iff its Killing form K is nondegenerate.*

Proof. The first direction follows easily; if $\mathfrak{g}$ is semisimple, $\operatorname{rad} K$ is a solvable ideal of $\mathfrak{g}$. But since the only solvable ideal of a semisimple Lie algebra is 0, we have $\operatorname{rad} K = 0$. The second direction requires slightly more work. Suppose that K is nondegenerate. Take I to be a solvable ideal of $\mathfrak{g}$. Now let n be the smallest number such that $I^n = 0$. Then, $a = B^{n-1}$ is obviously a nonzero ideal of $\mathfrak{g}$ and we can see that a is abelian. Then, for $x \in a$ and $y \in \mathfrak{g}$, the map $f = \operatorname{ad} x \operatorname{ad} y$ is a map from $\mathfrak{g}$ to a and $f^2 = 0$ as a is abelian. So, $K(x, y) = 0$ and $a \in \operatorname{rad} K$. Hence since K is nondegenerate we have $a = 0$, proving that $\mathfrak{g}$ is semisimple. ■

Throughout this section, we may have questioned how semisimple Lie algebras relate to simple Lie algebras. It turns out that semisimple Lie algebras are the direct sums of simple Lie algebras. We are unfortunately unable to prove this theorem as it requires some more advanced theory. Thus we defer it to [Bos12].

Theorem 4.9. *A Lie algebra $\mathfrak{g}$ is semisimple iff $\mathfrak{g} = \oplus_i \mathfrak{h}_i$ for simple Lie algebras $\mathfrak{h}_i$.*

Corollary 4.10. *A quotient of semisimple Lie algberas is semisimple. Furthermore, homomorphic images of semisimple Lie algebras are semisimple.*

We shall shift our focus away from the representation theory of abstract semisimple Lie algebras and look at the representations of $\mathfrak{sl}(2, \mathbb{C})$. Along the away, we will develop theory which will be greatly useful in the following sections.

Let $\phi : \mathfrak{sl}(n, \mathbb{C}) \to \mathfrak{gl}(V)$ be a representation of the special linear algebra. Consider the basis vector of $\mathfrak{sl}(2, \mathbb{C})$ $x = e_{11} - e_{22}$. Then, x acts diagonally on $\mathfrak{sl}(2, \mathbb{C})$ via the adjoint representation. By [Bos12], we have that $\phi(x)$ is diagonalizable. This implies that V is the direct sum of eigenspaces as follows:

$$V_\alpha = \{v \in V | x \cdot v = \alpha v\}$$

for eigenvalues $\alpha \in \mathbb{C}$. The eigenvalues α and the eigenspaces V_α weights and weight spaces respectively.

Proposition 4.11. *If $v \in V_\alpha$, then $e_{21}v \in V_{\alpha+2}$ and $e_{12}v \in V_{\alpha-2}$.*

Proof. Fortunatly, this proposition follows from direct computation. We have

$$\begin{aligned} x(e_{21}(v)) &= e_{21}(x(v)) + [x, e_{21}](v) \\ &= \alpha(x(v)) + 2x(v) \\ &= (\alpha + 2)(v). \end{aligned}$$

Similarly, we have

$$\begin{aligned} x(e_{12}(v)) &= e_{12}(x(v)) + [x, e_{12}](v) \\ &= \alpha(x(v)) - 2x(v) \\ &= (\alpha - 2)(v). \end{aligned}$$

■

Proposition 4.12. *The weights form a sequence $\alpha, \ldots, \alpha + 2n$ for some weight α.*

Proof. Fixing a weight α, we have that the vector space $\oplus_{n\in\mathbb{Z}} V_{a+2n}$ is invariant under $\mathfrak{g}$. Because V is irreducible, its only invariant subspaces are 0 and itself. Thus, the proposition follows. ■

As V is finite, there is some weight β that is larger than all other weights, making it a maximal weight. For some $v \in V_\beta$, we have $e_{21}v = 0$ and $e_{12}^n v \in V_{\beta-2n}$. As the difference between the weights in $\mathfrak{g}$ is two, we can take z to be the largest number such that $\beta - 2z$ is a weight.

Proposition 4.13. *The vector space $W = \operatorname{span}(v, e_{12}v, \ldots, e_{12}^z v)$ is invariant under $\mathfrak{g}$ and is a basis for V.*

Proof. Because $\mathfrak{g}$ is bilinear, we only need to show that W is invariant under the basis vectors of $\mathfrak{g}$. We have $e_{12}(e_{12}^i(v)) = e_{12}^{i+1}(v)$ so W is preserved under e_{12}. For $x = e_{11} - e_{22}$, we have $x(e_{12}^i(v)) = (\beta - 2i)e_{12}^i(v)$. To show invariance under e_{21}, we prove the following identity:

$$e_{21}(e_{12}^i(v)) = i(\beta - i + 1)e_{12}^{i-1}(v)..$$

We show this using induction on i. The base case is trivial. Assuming the identity holds for i, we have

$$\begin{aligned} e_{21}(e_{12}^{i+1}(v)) &= e_{21}e_{12}(e_{12}^i(v)) \\ &= ([e_{21}, e_{12}] + e_{12}e_{21})e_{12}^i(v) \\ &= x(e_{12}^i(v)) + (e_{12}e_{21}e_{12}^i(v)) \\ &= (\beta - 2i)e_{12}^i(v) + e_{12}(i(\beta - i + 1)e_{12}^{i-1}(v)) \\ &= (i + 1)(\beta - (i + 1) + 1)e_{12}^i(v). \end{aligned}$$

■

Thus, we may classify representations of $\mathfrak{sl}(2,\mathbb{C})$ by their maximal weight β. We have shown that all irreducible representations of $\mathfrak{g}$ can be written as $V = \oplus_{n=0}^{\beta} V_{2n-\beta}$. We may also wonder, given a representation of $\mathfrak{g}$, how can we find its maximal weight? We have

$$0 = e_{21}(e_{12}^{z+1}(v)) = (k+1)(\beta - k)e_{12}^{k}(v)$$

so β is simply the largest integer such that $\beta - 2k$ is a weight. From this, it also follows that if V is a representation of $\mathfrak{g}$ with decomposition $V = \oplus_n U_n$, then the number of irreducible U_i is simply the sum of the dimensions of the 0 and 1 weight spaces of the weight space decomposition of V.

5. Root Space Decompositions

In this section, we use theory from the previous sections, such as weight space decompositions, to provide an alternate decomposition for a Lie algebra: the root space decomposition. First, note that by Engel's theorem, a semisimple Lie algebra $\mathfrak{g}$ cannot consist of only nilpotent elements as that would cause $\mathfrak{g}$ to be nilpotent. So, $\mathfrak{g}$ contains semisimple elements. We have the existence of a subalagbera of $\mathfrak{g}$ consisting of only semisimple elements as the span of a semisimple element of $\mathfrak{g}$ is abelian.

Definition 5.1. A Lie subalgebra $\mathfrak{h}$ of a semisimple Lie algebra $\mathfrak{g}$ is a toral subalgebra if each element $x \in \mathfrak{h}$ is semisimple.

Lemma 5.2. *All toral subalgebras are abelian.*

Proof. For $x \in \mathfrak{h}, \operatorname{ad} x$ is diagonalizable. Because $\operatorname{ad} x$ is diagnolizable, if it has some nonzero eigenvalue a, then, there would be some nonzero $y \in \mathfrak{h}$ such that $\operatorname{ad} x(y) = ay$ which would imply that $\mathfrak{h}$ is not abelian. Thus, we show that $\operatorname{ad} x$ does not have any nonzero eigenvalues. Assume for the sake of contradiction that there is some $y \in \mathfrak{h}$ such that $[x, y] = ay$ for nonzero $a \in \mathbb{C}$. Then, $\operatorname{ad} y(x) = -ay$ and taking the $\operatorname{ad} y$ of both sides shows that $\operatorname{ad} y(\operatorname{ad} y(x)) = -a \operatorname{ad} y(y) = 0$. Since y is also diagonalizable, x is a linear combination of eigenvectors of $\operatorname{ad} y$. This implies that $\operatorname{ad} y$ must have a nonzero eigenvalue $\operatorname{ad} y(x) = -ay$ and we assumed a to be nonzero. But, we have $\operatorname{ad} y(\operatorname{ad} y(x)) = 0$ which is a contradiction. ∎

Definition 5.3. A maximal toral subalgebra is a Cartan subalgebra.

If W is a subspace of $\mathfrak{gl}(V)$ consisting of diagonalizable linear maps, then the elements of W are simultaneously diagonalizable, or in other words, there exists some invertible map S such that SxS^{-1} is diagonal for all $x \in W$. So, the action of a Cartan subalgebra $\mathfrak{h}$ is simultaneously diagonalizable. We then decompose $\mathfrak{g}$ into eigenspaces $\mathfrak{g}_\alpha = \{x \in \mathfrak{g} | [h, x] = \alpha(h)x\}$ for $h \in \mathfrak{h}$, $\alpha \in \mathfrak{h}^*$, where $\mathfrak{h}^*$ is the dual space, and $x \in \mathfrak{g}$. For the adjoint representation, we call the α roots and the $\mathfrak{g}_\alpha$ root spaces. Let R be the set of all roots. Then, we decompose $\mathfrak{g}$ as

$$\mathfrak{g} = C_{\mathfrak{g}}(\mathfrak{h}) \oplus_{\alpha \in R} \mathfrak{g}_\alpha.$$

Recalling $C_{\mathfrak{g}}(\mathfrak{h})$ as the centralizer, we clearly see that $\mathfrak{h} \subset C_{\mathfrak{g}}(\mathfrak{h})$. We produce a plethora of statements pertaining to this root space decomposition to build more theory.

Proposition 5.4.
(1) *If α and β are roots of $\mathfrak{g}$, then $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \in \mathfrak{g}_{\alpha+\beta}$.*
(2) *If $x \in \mathfrak{g}_\alpha$, then $\operatorname{ad} x$ is nilpotent.*
(3) *If $\alpha \neq \beta$ are roots pf $\mathfrak{g}$, mfg_α and $\mathfrak{g}_\beta$ are orthogonal under the Killing form.*

Proof. (1) For $x \in \mathfrak{g}_\alpha, y \in \mathfrak{g}_{beta}, h \in \mathfrak{h}$,

$$[h,[x,y]] = [[h,x],y] + [x,[h,y]] = (\alpha+\beta)(h)[x,y].$$

(2) First note that $\operatorname{ad} x$ sends V_γ to $V_{\gamma+\alpha}$. The result follows from part 1 and noting that the finite dimensionality of V implies that $V_{\gamma+k\alpha} = 0$ for some k. (3) The map $\operatorname{ad} x \operatorname{ad} y$ sends V_γ to $V_{\gamma+\alpha+\beta}$. The result follows from part 1 and noting that the finite dimensionality of V implies that $V_{\gamma+k(\alpha+\beta)} = 0$ for some k. ■

Recall that Cartan subalgebras are orthogonal to the root spaces of $\mathfrak{g}$. This observation, along with others, are used in [Bos12]. to prove the following theorem.

Theorem 5.5. *If $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$, then $C_{\mathfrak{g}}(\mathfrak{h}) = \mathfrak{g}$.*

We now consider the dual of a Cartan subalgebra. Specifically, isomorphisms between a Cartan subalgebra and its dual. For $h \in \mathfrak{h}$ take the functional $\alpha_h(x) = K(h,x)$. It follows from standard properties of the Killing form which we mentioned previously that $h \to \alpha_h(x)$ is an isomorphism. So, every functional $\alpha \in \mathfrak{h}^*$ has a unique (by the isomorphism) corresponding element $\delta \in \mathfrak{h}$ such that $\alpha(h) = K(\delta, h)$ for all $h \in \mathfrak{h}$. Thus we may define the Killing form on a dual space as follows:

$$K(\alpha,\beta) = K(\delta_\alpha, \delta_\beta).$$

It is somewhat abusive notation to have K denote the Killing form on both an algebra and its dual, but we do so anyways. We may prove some statemenets pertaining to $\mathfrak{h}^*$.

Proposition 5.6. *R spans $\mathfrak{h}^*$.*

Proof. Assume for the sake of contradiction that there exists some nonzero $h \in \mathfrak{h}$ such that $\alpha(h) = 0$ for all roots α. Then α acts trivially on each root space of $\mathfrak{g}$. But, $\mathfrak{g}$ is semisimple so its center is trivial. Thus, we have $h = 0$ which is a contradiction. ■

Proposition 5.7. *If α is a root then $-\alpha$ is also a root.*

Proof. Suppose that x_α is a nonzero vector in $\mathfrak{g}_\alpha$. Then, there is some y_α such that $K(x_\alpha, y_\alpha) \neq 0$. But if $alpha + \beta \neq 0$, $\mathfrak{g}_\alpha$ is orthogonal to $\mathfrak{g}_\beta$ with respect to the killing form so $y_\alpha \in \mathfrak{g}_{-\alpha}$. ■

Proposition 5.8. *The vector space $[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]$ is the 1 dimensional span of δ_α.*

Proof. For $x_{alpha} \in \mathfrak{g}_\alpha$ and $y_\alpha \in \mathfrak{g}_{-\alpha}$, we have

$$K(h,[x_{alpha}, y_\alpha]) = K([h,x_\alpha], y_\alpha) = \alpha(h)K(x_\alpha,y_\alpha) = K(h,\delta_\alpha)K(x_\alpha,y_\alpha)$$

for all $h \in \mathfrak{h}$. So, we have

$$K(h,[x_\alpha,y_\alpha] - K(x_\alpha, y_\alpha\delta_\alpha) = 0.$$

To conclude the proof, that each linear combination of a bracket $[x_\alpha, y_\alpha]$ is spanned by δ_α, we have $[x_\alpha, y_\alpha] = K(x_\alpha, y_\alpha)\delta_\alpha$ as K is nondegenerate. ■

Proposition 5.9. *For all roots $\alpha \in R, K(\alpha,\alpha) \neq 0$.*

Proof. As before, let $x_\alpha \in \mathfrak{g}_\alpha$ and $y_\alpha \in \mathfrak{g}_{-\alpha}$. However, this time we place the restriction that $K(x_\alpha, y_\alpha) \neq 0$. By the previous proposition, this spans a three dimensional subalgebra $\mathfrak{a}_\alpha$ of $\mathfrak{g}$. Now note that if the Killing form applied to $(\delta_\alpha, \delta_\alpha)$ is zero, then δ_α acts trivially on $\mathfrak{a}_\alpha$. Thus, we have $\mathfrak{a}_\alpha^1 = \mathbb{C}\delta_\alpha$ and $\mathfrak{a}_\alpha^2 = 0$. Thus, $\mathfrak{a}_\alpha$ is solvable. So, by Engel's theorem, $\operatorname{ad}\delta_\alpha$ is nilpotent. But, if δ_α is $\operatorname{ad}-$nilpotent, then it is impossible for δ_α to be a nonzero element of $\mathfrak{h}$. ■

Proposition 5.10. *For a root α and an element of the corresponding root space $x \in \mathfrak{g}_\alpha$, there exists some $y \in \mathfrak{g}_{-\alpha}$ such that* $\operatorname{span}(x_\alpha, y_\alpha, z_\alpha = [x_\alpha, y_\alpha])$ *is a is a subalgebra $\mathfrak{s}_\alpha$ of $\mathfrak{g}$ that is isomorphic to $\mathfrak{sl}_2(\mathbb{C})$.*

Proof. First, we have that $\dim(\mathfrak{s}_\alpha) = \dim(\mathfrak{sl}_2(\mathbb{C}))$ as we recall the standard basis for the special linear algebra of dimension two. Now, we have $\alpha(z_\alpha) \neq 0 \neq \alpha(\delta_\alpha)$. This allows us to scale y_α to have the same Lie brackets as the standard basis of $\mathfrak{sl}_2(\mathbb{C})$. For the explicit construction, we choose y_α to satisfy $K(x_\alpha, y_\alpha) = \frac{2}{K(\delta_\alpha,\delta_\alpha)}$. If we had $K(\delta_\alpha, \delta_\alpha) = 0$ this construction would clearly not work. From this construction, we have $z_\alpha + K(x_\alpha, y_\alpha)\delta_\alpha$. ∎

This isomorphism allows us to use the theory in the prior section on root spaces. Recall the sequence of roots for $\mathfrak{g}$ we found in the last section. These roots were symmetric on the origin. We consider a reflection in $\mathfrak{h}^*$ defined by $R_\alpha(\beta) = B - B(\mathfrak{z}_\alpha)\alpha$ for root α. We define the group generated by these reflections to be the Weyl group W.

Proposition 5.11. *The roots of $\mathfrak{g}$ are invariant with respect to W.*

Proof. To prove this, we consider roots congruent to the root β modulo α for root α. We show that these roots are invariant with respect to W_α. Consider the subalgebra $\mathfrak{s}_\alpha$ of $\mathfrak{g}$. Then the space $W = \oplus_{n\in\mathbb{Z}} V_{\beta+n\alpha}$ is a well-defined representation of $\mathfrak{s}_\alpha$. The sequence of roots of the root spaces is then

$$\beta + n\alpha, \ldots, \beta + m\alpha.$$

Then, the sequence of roots for $\mathfrak{s}_\alpha$ is

$$\beta(z_\alpha) + 2n, \ldots, \beta(z_\alpha) + 2m.$$

But, we remember that the roots are symmetric around the origin. So, this implies that $\beta(z_\alpha) = -m - n$ which is an integer. Now, we conclude the proof by showing that the considered roots are invariant with respect to the reflection. This is shown by the computation

$$W_\alpha(\beta + (n+k)\alpha) = \beta + (n+k)\alpha - (-(m+n) + 2(n-k)))\alpha = \beta + (m-k)\alpha.$$

∎

Corollary 5.12. *For roots $\alpha, \beta, \beta(z_\alpha) \in \mathbb{Z}$.*

We end this section with four more statements, the proofs of which can be found in [Bos12].

Proposition 5.13. *If α is a root, the only multiples of α which are also roots is $-\alpha$.*

Proposition 5.14. *Each root space $\mathfrak{g}_\alpha$ has dimension one.*

Proposition 5.15. *The Cartan subalgebra $\mathfrak{h}$ is spanned by the z_α and $\mathfrak{g}$ is generated by the root spaces $\mathfrak{g}_\alpha$.*

Theorem 5.16. *The Killing form K is positive definite on the roots and the subspace spanned by the roots in $\mathfrak{h}^*$.*

6. Root Systems

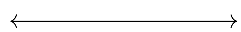
In this section, we consider a more geometric point of view by looking at root systems. Although they seem unrelated from Lie algebras at first, we show their usefulness in the study of Lie algebras.

Definition 6.1. A root system is a collection of roots R in Euclidian space E such that

(1) R is finite and spans E.
(2) If α is a root, then the only root multiples of α are $\pm\alpha$.
(3) The roots are invariant under the Weyl group.
(4) If α and β are roots, then
$$n_{\alpha\beta} = 2\frac{(\alpha, \beta)}{(\beta, \beta)}$$
is an integer for the standard inner product in E.

Suppose that for two roots α, β, we let θ be the angle between them. Then, we have $n_{\beta\alpha} = 2\cos(\theta)\frac{\|\beta\|}{\|\alpha\|}$. We now try and classify these roots. We take $l : E \to \mathbb{R}$ be a linear functional that is irrational with respect to the roots. The positive roots, R^+ are the roots $r \in R$ such that $l(r) > 0$. Similarly, the negative roots, R^- are the roots $r \in R$ such that $l(r) < 0$. A simple root is a positive root that is not the sum of two other positive roots.

Example. Consider the special linear algebra $\mathfrak{sl}_n(\mathbb{C})$. Recall the standard basis e_{ij} for $i \neq j$ and $h_{i,i+1} = e_{ii} - e_{i+1,i+1}$ for $1 \leq i \leq n-1$. Then, we consider $\mathfrak{h}$ to be the Cartan subalgebra defined by the span of all of the $h_{i,i+1}$. Let the functional corresponding to e_{kk} be $l_k : \mathfrak{sl}_n(\mathbb{C}) \to \mathbb{C}$. From this, we have that $\mathfrak{h}^*$ is the span of linear combinations $\sum_{i=0}^n a_i l_i$ for$\sum_{i=0}^n a_i = 0$. So, we have $[h_{ij}, e_{km}] = (l_k - l_m)h_{ij}e_{km}$ so the roots are $l_k - l_m$ for $1 \leq k \neq m \leq n$. Now, consider the function $L(\sum_{i=1}^n l_i a_i) = \sum_{i=1}^n a_i b_i$ for $\sum_{i=1}^n b_i = 0$ and $b_i > b_{i+1}$ for all i. From this, we see that the positive roots are $l_k - l_r$ for $k > r$. Within the positive roots, the simple positive roots are $L_n - L_{n-1}$. Pairs of consecutive simple roots have an angle of $\frac{2\pi}{3}$ between them and all other pairs of simple roots have angle $\pi/2$ between them. So, for $\mathfrak{sl}_2(\mathbb{C})$ we have the simple positive root $l_2 - l_1$. We visualize it as shown.



Proposition 6.2. *Suppose that α and β are roots where β is not a multiple of α. Then, the $\alpha-$string through β,*
$$\beta - p\alpha, \ldots, \beta - \alpha, \beta, \ldots, \beta + q\alpha$$
has at most 4 elements in a string. Furthermore, $p - q = n_{\beta\alpha}$.

Proof. Recalling the reflection W_α, we have $B - p\alpha = W_\alpha(p + q\alpha) = (\beta - n_{\beta\alpha}) - q\alpha$. So, we have $n_{\beta\alpha} = p - q$. And we have $|n_{\beta\alpha}| \leq 3$ which implies the second result. ∎

Corollary 6.3. *Take α and β to be roots as in the previous propsition. Then, $(\beta, \alpha) > 0$ iff $\alpha - \beta$ is a root. $(\beta, \alpha) < 0$ iff $\alpha + \beta$ is a root. $(\beta, \alpha) = 0$ if α and β are both roots or are both not roots.*

Proposition 6.4. *If α and β are simple roots such that $\alpha \neq \beta$, then $\pm(\alpha - \beta)$ are not roots.*

Proof. Recall that simple roots are the sums of two positive roots. So, if $\alpha - \beta$ is a root, then we must have that $\alpha = (\beta) + (\alpha - \beta)$ is not a simple root which is a contradiction. And if $\beta - \alpha$ is a root then we have that $\beta = (\alpha) + (\beta - \alpha)$ is not simple which is again a contradiction. ∎

Corollary 6.5. *If θ is the angle between two simple roots α, β, then $\theta \geq 90$.*

Proposition 6.6. *All of the simple roots are linearly independent.*

Proof. By our definition of a root system, we have that all of the roots are located on the same side of the hyperplane of the reflection in E. Combining this with the prior proposition that all angles between simple roots are not acute, we have our result. We have also shown that the root system is a basis in E. ■

Proposition 6.7. *Each simple positive root has a unique decomposition as a nonnegative linear combination of simple roots (where each coefficient of a simple root is an integer).*

Proof. Assume for the sake of contradicition that β is a positive root with a minimal $l(\beta)$, that is not equal to an integral nonnegative linear combination of simple roots. We can say that $\beta = \alpha + \zeta$ for positive roots α, ζ and the result follows from observing that $l(\gamma) < l(\beta)$. ■

Theorem 6.8. *If α_i is a set of simple roots in a root system, any positive root β has a decomposition as $\beta = \alpha_{i_1} + \ldots + \beta_{\alpha_{i_n}}$ where if $1 \leq m \leq n$ then $\alpha_{i_1} + \ldots + \alpha_{i_m}$ is positive.*

Proof. Obviously by the prior proposition we can write $B = \alpha_{i_1} + \ldots + \beta_{\alpha_{i_n}}$ as the simple roots form a basis. All that is left to show is that $1 \leq m \leq n \implies \alpha_{i_1} + \ldots + \alpha_{i_n}$ is positive. We show this with induction on m. We have $\beta = \alpha_{i_1} + \ldots + \alpha_{i_{m+1}}$. Then, we $(\beta, \alpha_{i_0}) + \ldots + (\beta, \alpha_{i=n}) = (\beta, \beta) > 0$ so $(\beta, \alpha_{i_k}) > 0$ for some α_{i_k}. Hence, we have $\beta - \alpha_{i_k}$ is a positive root by proposition ■

We have now built enough theory to discuss the relation between these root systems and the root space decomposition Lie algebras. We do this by considering taking the bracket of two root spaces of a Lie algebra $\mathfrak{g}$ with Cartan subalgebra $\mathfrak{h}$.

Lemma 6.9. *If $\alpha, \beta, \alpha+\beta$ are roots such that β is not a multiple of α, then $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = \mathfrak{g}_{\alpha+\beta}$.*

Proof. Each root space is one dimensional, so if we bracket $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta]$ we will get either 0 or $\mathfrak{g}_{\alpha+\beta}$. Thus, it will be sufficient to show that we will not get 0. Recalling $\mathfrak{s}_\alpha$, we construct the representation $W = \oplus_{n\in\mathbb{Z}}\mathfrak{g}_{\beta+n\alpha}$. We now see that $\mathfrak{h}$ is not a subspace of W and that each part of the decomposition of W has dimension one. Thus, we have W is irreducible. Thus, if $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = 0$, the sub representation $W = \oplus_{n\leq 0}\mathfrak{g}_{\beta+n\alpha}$ is a proper subrepresentation which contradicts W being irreducible. Thus, we have $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = \mathfrak{g}_{\alpha+\beta}$. ■

The beauty of this theorem lies in the observation that the root spaces of a root $\mathfrak{g}_\alpha$ generate all of the root spaces and thus the Lie algebra. One application of this theorem follows from considering maps between root spaces.

Definition 6.10. If, for Euclidian spaces E and E', $R \subset E$ and $R' \subset E'$ are root spaces, a linear map $\phi : R \to R'$ is an isomorphism if $(\phi(\alpha), \phi(\beta)) = (\alpha, \beta)$ for all roots $\alpha, \beta \in R$.

Theorem 6.11. *If $\mathfrak{g}$ and $\mathfrak{g}'$ are simple Lie algebras with cartan subalgebras $\mathfrak{h}$ and $\mathfrak{h}'$, with root directions given by the functionals l and l' respectively, then there exists an isomorphism between R and R' if there exists an isomoprphism between $\mathfrak{g}$ and $\mathfrak{g}'$.*

The proof of this theorem can be found in [Bos12].

Definition 6.12. A root system is reducible if there exists some $P \in R$ such that for $P' = R \setminus P$, all roots in P are orthogonal with all roots in P'. A root system is irreducible otherwise.

Theorem 6.13. *If $\mathfrak{g}$ is a simple Lie algebra, then its root system R is irreducible.*

Proof. Assume for the sake of contradiction hat R is reducible. Then, we have some subset $P \in R$ such that all roots in P are orthogonal with all roots in P'. Furthermore, P and P' are both nonempty. Take $\mathfrak{k}$ to be the subalgebra of $\mathfrak{g}$ that is spanned by all root systems $\mathfrak{g}_\alpha$ for root α. If β is some root in P', then $(\alpha+\beta,\alpha) \neq 0$ and $(\alpha+\beta,\beta) \neq 0$ so $\alpha+\beta$ is not a root, which implies from lemma that $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = 0$. Then, we have $[\mathfrak{k}, \mathfrak{g}_\beta] = 0$ which tells us that $\mathfrak{k}$ is a proper subalgebra of $\mathfrak{g}$ as $Z(\mathfrak{g}) = 0$. But, since $\mathfrak{k}$ is also an ideal, as $0 \in \mathfrak{k}$, we have that $\mathfrak{g}$ is not simple which is a contradiction. ■

7. The Classification Theorem

We first define the simple Lie algebras. The simple Lie algebras are a family of four types of Lie algebras: A_n, B_n, C_n, D_n. All simple Lie algebras are either in this family or in the family of exceptional Lie algebras: E_6, E_7, E_8, F_4, G_2.

Definition 7.1. A_n : The algebra $\mathfrak{sl}(n+1,\mathbb{F})$ of endomorphisms of V with trace 0.

Definition 7.2. B_n : We let V have dimension $2n+1$ and take f to be a nondegenerate symmetric bilinear form on V with matrix $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & I_n \\ 0 & I_n & 0. \end{bmatrix}$ Then we define the orthogonal algebra $\mathfrak{o}(2n+1,\mathbb{F})$to be all endomorphisms x of V such that $f(x(v),w) = -f(v,x(w))$ for $w \in V$.

Definition 7.3. C_n : We let V have dimension $2n$ and take f to be a nondegenerate skew-symmetric form on V with matrix $\begin{bmatrix} 0 & I_n \\ -I_n & 0. \end{bmatrix}$Then we define the symplectic algebra $\mathfrak{sp}(2n,\mathbb{F})$ to be all endomorphisms x of V such that $f(x(v),w) = -f(v,x(w))$.

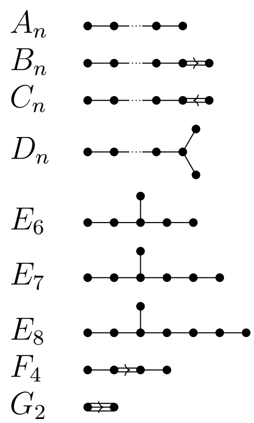
Definition 7.4. $D_{n\geq 2}$: We let V have dimension $2n$ and define f to be a nondegenerate symmetric bilinear form on V with matrix $\begin{bmatrix} 0 & I_n \\ I_n & 0. \end{bmatrix}$ Then we define norther orthogonal algebra $\mathfrak{o}(2n,\mathbb{F})$ to be all endomorphisms x of V such that $f(x(v),w) = -f(v,x(w))$.

As we have constructed the simple Lie algebras, we can move on to Dynkin diagrams and the classification theorem.

Definition 7.5. The Dynkin diagram of a root system is created by drawing a node $\circ$ for each simple root and joining two roots α and β with $-n_{\beta\alpha}$.

Furthermore, the edges of a Dynkin diagram are directed to denote root size.If the Dynkin diagram corresponding to a root system is connected, the system is irreducible. And because each Dynkin diagram corresponds to a unique root system of a semisimple Lie algebra, we determine the possible connected Dynkin diagrams for semisimple Lie algebra $\mathfrak{g}$. Before we classify the Dynkin diagrams of semisimple Lie algebras, we view an easier analogue of Dynkin diagrams, Coexter diagrams. A Coexter diagram with n nodes corresponds to the system $(e_1,\ldots,e_n)$ of linearly independent unit vectors in E. The angle between two of these vectors is $\frac{k\pi}{k+1}$ for $1 \leq k \leq 5$. And since these vectors are unit vectors, we have $(e_i,e_j) = \cos(\theta)\|e_i\|\|e_j\| = \cos(\theta)$. We call a diagram admissable if it corresponds to such a system. For any admissable diagram, directing the arrows produces a Dynkin diagram as one can scale the vectors to have the relation in size required by the directions.

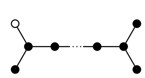
Theorem 7.6. *All complex semisimple Lie algebras have one of the following Dynkin diagrams:*



Remark 7.7. Although we were able to define the simple Lie algebras, there is no way to construct the exceptional Lie algebras E_6, E_7, E_8, F_4, G_2 besides stating their Dynkin diagrams. Although we do not prove so, the Dynkin diagrams shown are the ones corresponding with the family of simple Lie algebras denoted on the left.

We state the key details that when proved, are able to prove this classification theorem. Let N be an admissable diagram with n nodes.

(1) A subdiagram of N, created by removing nodes and the lines connected to those nodes is admissable.
(2) The number of pairs of connected nodes is at most $n-1$.
(3) N does not have any loops.
(4) No node has a degree greater than 3.
(5) The only admissable diagram with a pair of nodes with three lines between them is the Coexter diagram corresponding with G_2.
(6) Any string of nodes connected to each other by one line with only the ends of the string connected to other nodes can be collapsed into one node to create an admissable diagram.
(7) N does not admit any subdiagram of a string of nodes where the endpoints are connected with two lines to their adjacent node and all other nodes are connected by one line to their adjacent nodes (besides the adjacent nodes to the endpoints).
(8) N does not admit any subdiagram that is constructed from the Coexter diagram of D_n with two lines between the left-most node and its adjacent.
(9) N does not admit any subdiagram of the form with the upper left node filled in as



well.
(10) The Coexter diagram of a string of 5 nodes with one line connecting each node except for two lines connecting the second and third node is not admissable.

A full proof may be found in [HUM94].

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