There is One Separable Hilbert Space (of Each Dimension)

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Introduction

Linear algebra, which studies vector spaces and their properties, offers many mathematical insights into these spaces. However, many results of linear algebra do not apply to all vector spaces.

In this presentation, we will introduce various concepts from functional analysis, a field of math which generalizes many results from linear algebra.

Vector Spaces

Definition (Vector Space)

A *vector space* V over a field $\mathbb{F}$ is a set (whose elements are called vectors), together with the associative and commutative operation of addition of vectors, and the associative and distributive operation of multiplication of vectors by elements of $\mathbb{F}$ (called scalars).

Vector Spaces

Example (Euclidean Space $\mathbb{R}^n$)

The space $\mathbb{R}^n$ is a vector space under componentwise addition and multiplication by real scalars. Vectors in $\mathbb{R}^n$ are n-tuples $(x_1, \ldots, x_n)$, where $x_1, \ldots, x_n \in \mathbb{R}$.

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Example (Unitary Space $\mathbb{C}^n$)

Similarly, $\mathbb{C}^n$ under componentwise addition and scalar multiplication is also a vector space. Vectors $\mathbb{C}^n$ are n-tuples $(x_1, \ldots, x_n)$, where $x_1, \ldots, x_n \in \mathbb{C}$.

Vector Spaces

Example (Sequence Space ℓ^∞)

The space of all bounded sequences under termwise addition and scalar multiplication, denoted ℓ^∞, is a vector space. Vectors in this space are sequences $(x_n) = (x_1, x_2, x_3, \ldots)$, where $x_n \in \mathbb{C}$.

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Example (Function Space $\mathcal{P}[a, b]$)

The space of real-valued polynomials under pointwise addition and scalar multiplication, denoted $\mathcal{P}[a, b]$, is a vector space. Vectors in this space are polynomials with real coefficients.

Vector Spaces

Definition (Basis)

A *basis* of a vector space V is the smallest set of vectors B such that each element of V can be written in the form

$$a_1 v_1 + \cdots + a_n v_n$$

for some scalars $a_1, \ldots, a_n$ and vectors $v_1, \ldots, v_n \in B$.

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All bases of a finite-dimensional vector space have the same number of vectors. Intuitively, this makes sense because bases allow us to creates coordinates for a vector space via the coefficients $a_1, \ldots, a_n$ as above.

Vector Spaces

Definition (Dimension)

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Here are the dimensions of the spaces mentioned before:

$$\dim \mathbb{R}^n = n,$$
$$\dim \mathbb{C}^n = n,$$
$$\dim \ell^\infty = \infty,$$
$$\dim \mathcal{P}[a, b] = \infty.$$

Vector Spaces

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Theorem

A finite-dimensional vector space V over a field $\mathbb{F}$, with $\dim V = n$, is isomorphic to $\mathbb{F}^n$.

Vector Spaces

Proof.

Choose a basis $v_1, \ldots, v_n$ of V. Then we can write any vector $v \in V$ as

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Now take a basis $e_1, \ldots, e_n$ of $\mathbb{F}^n$. Then there exists a unique element $x \in \mathbb{F}^n$ where

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We will spend the rest of the talk developing a way to generalize this theorem to infinite-dimensional spaces.

Inner Products

Definition (Inner product)

An *inner product space* is a vector space V equipped with an *inner product* written $\langle \cdot, \cdot \rangle$ that satisfies the following properties for all $x, y, z \in V$ and scalars α:

$$\langle x + y, z \rangle = \langle x, z \rangle + \langle y, z \rangle \qquad \langle \alpha x, y \rangle = \alpha \langle x, y \rangle$$

$$\langle x, y \rangle = \overline{\langle y, x \rangle}$$

$$\langle x, x \rangle \geq 0 \qquad \langle x, x \rangle = 0 \iff x = 0$$

Inner Products

Example ($\mathbb{R}^n$, $\mathbb{C}^n$)

On $\mathbb{R}^n$ and $\mathbb{C}^n$, we define the standard inner product (also known as the dot product) for vectors $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_n)$ by

$$\langle x, y \rangle = \sum_{i=1}^{n} x_i \overline{y_i}.$$

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Example ($\mathcal{P}[a, b]$)

On $\mathcal{P}[a, b]$, we define an inner product for vectors $x = x(t)$, $y = y(t)$ by

$$\langle x, y \rangle = \int_a^b x(t) y(t)\, dt.$$

Inner Products

Note that we cannot extend our definition of the standard inner product on $\mathbb{C}^n$ to ℓ^∞.
For example, if we tried to find the dot product of the sequence

$$x = (1, 1, 1, \ldots)$$

with itself, we would get

$$\langle x, x \rangle = \sum_{i=1}^{\infty} 1,$$

which doesn't converge.
To fix this issue, we will impose a further restriction on the sequences we allow.

Inner Products

Definition (Sequence space ℓ^2)

The sequence space $\ell^2 \subseteq \ell^\infty$ is defined to be the space of all sequences $(x_n) = (x_1, x_2, x_3, \ldots)$, where $x_n \in \mathbb{C}$, such that

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converges.

The space ℓ^2, under termwise addition and scalar multiplication, is a vector space. Furthermore, this space is an inner product space with the inner product of vectors $x = (x_n)$ and $y = (y_n)$ being

$$\langle x, y \rangle = \sum_{n=1}^{\infty} x_n \overline{y_n}.$$

Hilbert Spaces

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Example

The space $\mathbb{Q}^n$ is dense in $\mathbb{R}^n$ and thus $\mathbb{R}^n$ is separable. By a similar argument, $\mathbb{C}^n$ is separable.

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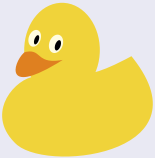
Corollary

We have



as a clear result of our theorem.

Thank you for your time.