Pell's Equation

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July 8, 2022

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Example

Let d be equal to 2. Thus,

$$3^2 - 2.2^2 = 1$$

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Example

Let d be equal to 6. Thus,

$$3^2 - 6.1^2 = 3$$

Fundamental Solution

Definition

Fundamental solution refers to any solution which can solve one or more *root causes*. That is, the root of the problem is used to construct theorems and problems based on them.

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Example

Let d be equal to 3. Thus,

$$x^2 - 3y^2 = 1$$

$$x^2 = 3y^2 + 1$$

The smallest solution that satisfies is (2,1); thus, it is the *fundamental solution* of this equation.

Continued Fraction

Definition

Continued Fraction is any fraction whose numerator is an integer and denominator is a quantity plus a fraction, and the fraction's numerator and denominator follow a similar pattern.

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1. highly important to understand various different topics in math, including but not limited to *Pell's Equation*
2. has many types, such as *Finite Continued Fraction* or *Generalized Continued Fraction*
3. best way to solve *Pell's Equation* and get best approximations for irrational numbers.

Representation of Continued Fractions

Representing *Continued Fractions* is indeed a fun but mind-blowing process. Well, because there are a lot of different ways to do it!

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{\ddots}}}}$$

or

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots + \cfrac{1}{a_{n-1} + \cfrac{1}{a_n}}}}}$$

Representation of Continued Fraction

There are *abbreviated notation* representations as well!

$$[a_0; a_1, a_2, a_3, \ldots]$$

or

$$[a_0; a_1, a_2, \ldots, a_{n-1}, a_n]$$

And, a site note is that it all depends on whether represent a rational number or irrational number. That is, whether they are *Finite Continued Fraction* or *Infinite Continued Fraction*.

Solutions to Continued Fraction

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$$4 = \boxed{4}.1 + 0$$

Some Interesting Examples to Continued Fraction

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$$\pi = \cfrac{4}{1 + \cfrac{1^2}{2 + \cfrac{3^2}{2 + \cfrac{5^2}{2 + \ddots}}}}$$

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Continued Fractions are a mystery itself since they can give various different representations to π.

$$\pi = \cfrac{4}{1 + \cfrac{1^2}{2 + \cfrac{3^2}{2 + \cfrac{5^2}{2 + \ddots}}}}$$

or

$$\pi = 3 + \cfrac{1^2}{6 + \cfrac{3^2}{6 + \cfrac{5^2}{6 + \cfrac{7^2}{6 + \ddots}}}}$$

Some Interesting Examples to Continued Fraction

or, even

$$\pi = \cfrac{4}{1 + \cfrac{1^2}{3 + \cfrac{2^2}{5 + \cfrac{3^2}{7 + \ddots}}}}$$

or, again, even

Some Interesting Examples to Continued Fraction

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$$\pi = \cfrac{4}{1 + \cfrac{1^2}{3 + \cfrac{2^2}{5 + \cfrac{3^2}{7 + \ddots}}}}$$

or, again, even

$$\pi = 3 + \cfrac{1}{7 + \cfrac{1}{15 + \cfrac{1}{1 + \cfrac{1}{292 + \cfrac{1}{1 + \ddots}}}}}$$

Convergents

Definition

Maybe the most important asset to solve the *Pell's Equation*, *convergents* derive from *Continued Fractions*. And, they are what we get when we truncate a continued fraction after some number of terms. In *Continued Fraction*, *convergents* are the best approximation of that number.

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1. actively used to find the *fundamental solution* and other solutions to *Pell's Equation*, especially when trial-and-error does not work.

Why?

Imagine you are trying to find the *fundamental solution* to the *Pell's Equation* in which d equals to 109. Well, then, you need to try until when x equals to 158070671986249 and y equals to 15140424455100.

Convergents

Example

The *convergents* of

$$\frac{93}{17} = 5 + \cfrac{1}{2 + \cfrac{1}{8}}$$

are [5], [5;2], and [5;2,8]

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are [5], [5;2], and [5;2,8]

However, even the *convergents* are as interesting as the *Continued Fractions* themselves. That is, you can find the *convergents* of $\sqrt{2}$ through the method

$$\frac{numerator + (2.denominator)}{numerator + denominator}$$

Solutions to Pell's Equation

Theorem 4.1

By changing the value of y, then basing the value of x on it, we can find the *fundamental solutions* to the most basic *Pell's Equations*.

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Suppose d is a positive small integer such as 2, then we can solve it easily through trial-and-error method.

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Suppose d is a positive small integer such as 2, then we can solve it easily through trial-and-error method.

Proof

$$x^2 - (2.1^2) = 1$$
$$x^2 - 2 = 1$$
$$x^2 = 2 + 1 = 3$$
$$x = \sqrt{3}$$

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Proof

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$$x^2 - 2 = 1$$
$$x^2 = 2 + 1 = 3$$
$$x = \sqrt{3}$$

$$x^2 - (2.2^2) = 1$$
$$x^2 - 8 = 1$$
$$x^2 = 9$$
$$x = 3$$

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The *fundamental solution* of a *Pell's Equation* can be found by finding and and testing each consecutive *convergents* of $\sqrt{d}$ until a solution is found.

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Suppose d is a greater number than 2, such as 6. Before, let's identify the *convergents* of $\sqrt{6}$.

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$$\frac{2}{1}, \frac{5}{2}, \frac{22}{9}, \frac{49}{20}$$

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Proof

$$2^2 - 6.1^2 = -2$$

$$5^2 - 6.2^2 = 1$$

Solutions to Pell's Equation

Theorem 4.3

By finding the convergents of $\sqrt{d}$ through Continued Fractions and applying the convergents to *Pell's Equation*, other solutions to the Pell's Equation can also be found.

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$$22^2 - 6.9^2 = -2$$

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$$\frac{2}{1}, \frac{5}{2}, \frac{22}{9}, \frac{49}{20}$$

Proof

$$22^2 - 6.9^2 = -2$$

$$49^2 - 6.20^2 = 1$$

Solutions to Pell's Equation

Theorem 4.4

By using the equation $(x + y\sqrt{d})^n = \alpha^n$, raising n to different powers, and then applying the fundamental solution, we can find the other solutions to Pell's Equation.

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Since it is the smallest value that d can get, suppose d is equal to 2.

Proof

$$x^2 - 2y^2 = 1$$

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Proof

$$x^2 - 2y^2 = 1$$

$$x + \sqrt{2}y = \alpha$$

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Since it is the smallest value that d can get, suppose d is equal to 2.

Proof

$$x^2 - 2y^2 = 1$$

$$x + \sqrt{2}y = \alpha$$

$$(x + \sqrt{2}y)^n = \alpha^n$$

Solutions to Pell's Equation

Proof

$$(3 + 2\sqrt{2})^n = \alpha^n$$

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$$(3+2\sqrt{2})^n = \alpha^n$$

$(3+2\sqrt{2})^2 = \alpha^2$

Solutions to Pell's Equation

Proof

$$(3+2\sqrt{2})^n = \alpha^n$$

$$(3+2\sqrt{2})^2 = \alpha^2$$

$$9+12\sqrt{2}+8 = 17+12\sqrt{2} = \alpha$$

Solutions to Pell's Equation

Proof

$$(3 + 2\sqrt{2})^n = \alpha^n$$

$$(3 + 2\sqrt{2})^2 = \alpha^2$$

$$9 + 12\sqrt{2} + 8 = 17 + 12\sqrt{2} = \alpha$$

$$17^2 - 2.12^2 = 1$$

Solutions to Pell's Equation

Proof

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$$(3+2\sqrt{2})^2 = \alpha^2 \qquad (3+2\sqrt{2})^3 = \alpha^3$$

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Proof

$$(3+2\sqrt{2})^n = \alpha^n$$

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$$9+12\sqrt{2}+8 = 17+12\sqrt{2} = \alpha$$

$$17^2 - 2.12^2 = 1$$

$$(3+2\sqrt{2})^3 = \alpha^3$$

$$27+54\sqrt{2}+72+16\sqrt{2} = 99+70\sqrt{2}$$

Solutions to Pell's Equation

Proof

$$(3+2\sqrt{2})^n = \alpha^n$$

$$(3+2\sqrt{2})^2 = \alpha^2$$

$$9+12\sqrt{2}+8 = 17+12\sqrt{2} = \alpha$$

$$17^2 - 2.12^2 = 1$$

$$(3+2\sqrt{2})^3 = \alpha^3$$

$$27+54\sqrt{2}+72+16\sqrt{2} = 99+70\sqrt{2}$$

$$99^2 - 2.70^2 = 1$$

Solutions to Generalized Pell's Equation

Theorem 4.5

To find the other solutions to Generalized Pell's Equation, we first need to find its fundamental solution. Then, we need to make n=1 and find that equation's fundamental solution. Then, we need to apply their fundamental solutions to the equation $(x + y\sqrt{d})^n = \alpha^n$ and raise their powers. Lastly, we need to multiply both of them.

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Suppose d equals to 6 and n equals to 3.

Proof

$$x^2 - 6y^2 = 3$$

$$x^2 = 6y^2 + 3$$

$$x^2 = 6.1^2 + 3 = 9$$

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Suppose d equals to 6 and n equals to 3.

Proof

$$x^2 - 6y^2 = 3$$

$$x^2 = 6y^2 + 3$$
$$x^2 = 6.1^2 + 3 = 9$$

$$x = 3$$

Solutions to Generalized Pell's Equation

Proof

Thus, the *fundamental solution* to our *Generalized Pell's Equation* is (3,1).

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We already know the *fundamental solution* to $x^2 - 6y^2 = 1$ which is (5,2). Thus,

Solutions to Generalized Pell's Equation

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$$x^2 - 6y^2 = 1$$

We already know the *fundamental solution* to $x^2 - 6y^2 = 1$ which is (5,2). Thus,

$$3 + 1\sqrt{6} = \alpha$$

$$5 + 2\sqrt{6} = \beta$$

Solutions to Generalized Pell's Equation

Proof

$$\alpha.\beta = (3 + 1\sqrt{6}).(5 + 2\sqrt{6})$$

Solutions to Generalized Pell's Equation

Proof

$$\alpha.\beta = (3 + 1\sqrt{6}).(5 + 2\sqrt{6})$$

$$\alpha.\beta = 15 + 5\sqrt{6} + 6\sqrt{6} + 12 = 27 + 11\sqrt{6}$$

Solutions to Generalized Pell's Equation

Proof

$$\alpha.\beta = (3 + 1\sqrt{6}).(5 + 2\sqrt{6})$$

$$\alpha.\beta = 15 + 5\sqrt{6} + 6\sqrt{6} + 12 = 27 + 11\sqrt{6}$$

Now, we have (27,11). Thus,

Solutions to Generalized Pell's Equation

Proof

$$\alpha.\beta = (3 + 1\sqrt{6}).(5 + 2\sqrt{6})$$

$$\alpha.\beta = 15 + 5\sqrt{6} + 6\sqrt{6} + 12 = 27 + 11\sqrt{6}$$

Now, we have (27,11). Thus,

$$27^2 - 6.11^2 = 3$$

Thank You

Thank you for your attention!