Pell's Equation

Adanur Nas

July 8, 2022

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Pell's Equation

The Equation

$$x^2 - dy^2 = 1$$

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$$x^2 - dy^2 = 1$$

1. d is a positive integer but not a perfect square

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2. x and y are integers

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Example

Let d be equal to 2. Thus,

$$3^2 - 2.2^2 = 1$$

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1. similar to the Pell's Equation except for its solution

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The Equation

 $x^2 - dy^2 = n$

- 1. similar to the Pell's Equation except for its solution
- 2. *n* can be any integer except 1
- 3. uses Pell's Equation to provide solutions

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Example

Let d be equal to 6. Thus,

$$3^2 - 6.1^2 = 3$$

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Fundamental Solution

Definition

Fundamental solution refers to any solution which can solve one or more *root causes*. That is, the root of the problem is used to construct theorems and problems based on them.

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1. highly important to solve the *Pell's Equation* since every solution method is based on it

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1. highly important to solve the *Pell's Equation* since every solution method is based on it

Example

Let d be equal to 3. Thus,

$$x^2 - 3y^2 = 1$$

$$x^2 = 3y^2 + 1$$

The smallest solution that satisfies is (2,1); thus, it is the *fundamental* solution of this equation.

Continued Fraction is any fraction whose numerator is an integer and denominator is a quantity plus a fraction, and the fraction's numerator and denominator follow a similar pattern.

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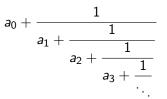
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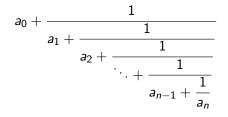
3. best way to solve *Pell's Equation* and get best approximations for irrational numbers.

Representation of Continued Fractions

Representing *Continued Fractions* is indeed a fun but mind-blowing process. Well, because there are a lot of different ways to do it!



or



There are abbreviated notation representations as well!

 $[a_0; a_1, a_2, a_3, \ldots]$

or

$$[a_0; a_1, a_2, \ldots, a_{n-1}, a_n]$$

And, a site note is that it all depends on whether represent a rational number or irrational number. That is, whether they are *Finite Continued Fraction* or *Infinite Continued Fraction*.

Solutions to Continued Fraction

Continued Fractions can be solved through many different methods. But, one of them is particularly easy and fun to do.

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The Continued Fraction of $\frac{149}{17}$ is

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The Continued Fraction of $\frac{149}{17}$ is $149 = \boxed{8}.17 + 13$

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The Continued Fraction of $\frac{149}{17}$ is $149 = \boxed{8}.17 + 13$ $17 = \boxed{1}.13 + 4$

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The Continued Fraction of $\frac{149}{17}$ is 149 = 8.17 + 13 17 = 1.13 + 413 = 3.4 + 1

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Continued Fractions are a mystery itself since they can give various different representations to π .

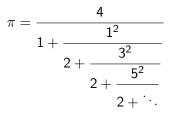
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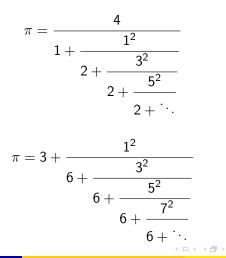
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Image: A matrix and a matrix

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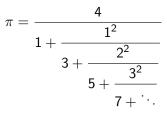


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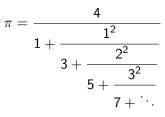
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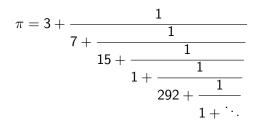
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or, even



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Definition

Maybe the most important asset to solve the *Pell's Equation, convergents* derive from *Continued Fractions*. And, they are what we get when we truncate a continued fraction after some number of terms. In *Continued Fraction, convergents* are the best approximation of that number.

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1. actively used to find the *fundamental solution* and other solutions to *Pell's Equation*, especially when trial-and-error does not work.

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1. actively used to find the *fundamental solution* and other solutions to *Pell's Equation*, especially when trial-and-error does not work.

Why?

Imagine you are trying to find the *fundamental solution* to the *Pell's Equation* in which d equals to 109. Well, then, you need to try until when x equals to 158070671986249 and y equals to 15140424455100.

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Example

The convergents of

$$\frac{93}{17} = 5 + \frac{1}{2 + \frac{1}{8}}$$

are [5], [5;2], and [5;2,8]

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Convergents

Example

The convergents of

$$\frac{93}{17} = 5 + \frac{1}{2 + \frac{1}{8}}$$

are [5], [5;2], and [5;2,8]

However, even the *convergents* are as interesting as the *Continued Fractions* themselves. That is, you can find the *convergents* of $\sqrt{2}$ through the method

 $\frac{numerator + (2.denominator)}{numerator + denominator}$

Theorem 4.1

By changing the value of y, then basing the value of x on it, we can find the *fundamental solutions* to the most basic *Pell's Equations*.

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Image: A matrix and A matrix

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Suppose d is a positive small integer such as 2, then we can solve it easily through trial-and-error method.

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$$x^{2} - (2.1^{2}) = 1$$
$$x^{2} - 2 = 1$$
$$x^{2} = 2 + 1 = 3$$
$$x = \sqrt{3}$$

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$$x^{2} - (2.1^{2}) = 1$$

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$$x^{2} - 8 = 1$$

$$x^{2} = 2 + 1 = 3$$

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Theorem 4.2

The *fundamental solution* of a *Pell's Equation* can be found by finding and and testing each consecutive *convergents* of \sqrt{d} until a solution is found.

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Suppose *d* is a greater number than 2, such as 6. Before, let's identify the *convergents* of $\sqrt{6}$.

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$$\frac{2}{1}, \frac{5}{2}, \frac{22}{9}, \frac{49}{20}$$

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$$2^2 - 6.1^2 = -2$$

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Proof

$$2^2 - 6.1^2 = -2$$

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 $5^2 - 62^2 = 1$

Theorem 4.3

By finding the convergents of \sqrt{d} through Continued Fractions and applying the convergents to *Pell's Equation*, other solutions to the Pell's Equation can also be found.

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Since we already know the first *convergents* of $\sqrt{6}$, suppose *d* is equal to 6.

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Since we already know the first *convergents* of $\sqrt{6}$, suppose *d* is equal to 6.

$$\frac{2}{1}, \frac{5}{2}, \frac{22}{9}, \frac{49}{20}$$

$$22^2 - 6.9^2 = -2$$

$$49^2 - 6.20^2 = 1$$

Theorem 4.4

By using the equation $(x + y\sqrt{d})^n = \alpha^n$, raising n to different powers, and then applying the fundamental solution, we can find the other solutions to Pell's Equation.

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Since it is the smallest value that d can get, suppose d is equal to 2. Proof

$$x^2 - 2y^2 = 1$$

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$$x^2 - 2y^2 = 1$$

$$x + \sqrt{2}y = \alpha$$

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Since it is the smallest value that d can get, suppose d is equal to 2. Proof

$$x^2 - 2y^2 = 1$$

$$x + \sqrt{2}y = \alpha$$

$$(x+\sqrt{2}y)^n = \alpha^n$$

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$$(3+2\sqrt{2})^n = \alpha^n$$

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$$(3+2\sqrt{2})^n = \alpha^n$$

$$(3+2\sqrt{2})^2 = \alpha^2$$

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Proof

$$(3+2\sqrt{2})^n = \alpha^n$$

$$(3+2\sqrt{2})^2 = \alpha^2$$

$$9 + 12\sqrt{2} + 8 = 17 + 12\sqrt{2} = \alpha$$

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$$(3+2\sqrt{2})^n = \alpha^n$$

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$$(3 + 2\sqrt{2})^2 = \alpha^2$$
$$9 + 12\sqrt{2} + 8 = 17 + 12\sqrt{2} =$$

$$17^2 - 2.12^2 = 1$$

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Proof

$$(3+2\sqrt{2})^n = \alpha^n$$

$$(3 + 2\sqrt{2})^2 = \alpha^2 \qquad (3 + 2\sqrt{2})^3 = \alpha^3$$
$$9 + 12\sqrt{2} + 8 = 17 + 12\sqrt{2} = \alpha$$
$$17^2 - 2.12^2 = 1$$

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Proof

$$(3+2\sqrt{2})^n = \alpha^n$$

$$(3 + 2\sqrt{2})^2 = \alpha^2 \qquad (3 + 2\sqrt{2})^3 = \alpha^3$$
$$9 + 12\sqrt{2} + 8 = 17 + 12\sqrt{2} = \alpha \qquad 27 + 54\sqrt{2} + 72 + 16\sqrt{2} = 99 + 70\sqrt{2}$$
$$17^2 - 2.12^2 = 1$$

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Proof

$$(3+2\sqrt{2})^n = \alpha^n$$

$$(3 + 2\sqrt{2})^2 = \alpha^2 \qquad (3 + 2\sqrt{2})^3 = \alpha^3$$
$$9 + 12\sqrt{2} + 8 = 17 + 12\sqrt{2} = \alpha \qquad 27 + 54\sqrt{2} + 72 + 16\sqrt{2} = 99 + 70\sqrt{2}$$
$$17^2 - 2.12^2 = 1 \qquad 99^2 - 2.70^2 = 1$$

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Theorem 4.5

To find the other solutions to Generalized Pell's Equation, we first need to find its fundamental solution. Then, we need to make n=1 and find that equation's fundamental solution. Then, we need to apply their fundamental solutions to the equation $(x + y\sqrt{d})^n = \alpha^n$ and raise their powers. Lastly, we need to multiply both of them.

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Suppose d equals to 6 and n equals to 3.

Theorem 4.5

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Suppose d equals to 6 and n equals to 3.

Proof

$$x^2 - 6y^2 = 3$$

$$x^2 = 6y^2 + 3$$

 $x^2 = 6.1^2 + 3 = 9$

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Theorem 4.5

To find the other solutions to Generalized Pell's Equation, we first need to find its fundamental solution. Then, we need to make n=1 and find that equation's fundamental solution. Then, we need to apply their fundamental solutions to the equation $(x + y\sqrt{d})^n = \alpha^n$ and raise their powers. Lastly, we need to multiply both of them.

Suppose d equals to 6 and n equals to 3.

Proof

$$x^2 - 6y^2 = 3$$

$$x^2 = 6y^2 + 3$$
$$x^2 = 61^2 + 3 = 9$$

x = 3

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Proof

Thus, the fundamental solution to our Generalized Pell's Equation is (3,1).

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Proof

Thus, the *fundamental solution* to our *Generalized Pell's Equation* is (3,1). Now, we will make the equation equal to 1.

$$x^2 - 6y^2 = 1$$

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Proof

Thus, the fundamental solution to our Generalized Pell's Equation is (3,1). Now, we will make the equation equal to 1.

$$x^2 - 6y^2 = 1$$

We already know the *fundamental solution* to $x^2 - 6y^2 = 1$ which is (5,2). Thus,

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Proof

Thus, the fundamental solution to our Generalized Pell's Equation is (3,1). Now, we will make the equation equal to 1.

$$x^2 - 6y^2 = 1$$

We already know the *fundamental solution* to $x^2 - 6y^2 = 1$ which is (5,2). Thus,

$$3 + 1\sqrt{6} = \alpha$$
$$5 + 2\sqrt{6} = \beta$$

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$$\alpha.\beta = (3 + 1\sqrt{6}).(5 + 2\sqrt{6})$$

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Proof

$$\alpha.\beta = (3 + 1\sqrt{6}).(5 + 2\sqrt{6})$$
$$\alpha.\beta = 15 + 5\sqrt{6} + 6\sqrt{6} + 12 = 27 + 11\sqrt{6}$$

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Proof

$$\alpha.\beta=(3+1\sqrt{6}).(5+2\sqrt{6})$$

$$\alpha.\beta=15+5\sqrt{6}+6\sqrt{6}+12=27+11\sqrt{6}$$
 Now, we have (27,11). Thus,

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Proof

$$\alpha.\beta = (3+1\sqrt{6}).(5+2\sqrt{6})$$

$$\alpha.\beta = 15+5\sqrt{6}+6\sqrt{6}+12 = 27+11\sqrt{6}$$
 Now, we have (27,11). Thus,

$$27^2 - 6.11^2 = 3$$

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Thank you for your attention!

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