PELL'S EQUATION

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Abstract. In this paper, we introduced and detailed the *Pell's Equation* and its types. Moreover, in order to provide solutions to *Pell's Equation*, we also introduced and detailed *Continued Fraction* and its types. Since there are not enough accessible papers on both topics, we aimed to provide articulable solutions and explanations to *Pell's Equation*. Therefore, in this paper, the readers will find the answers to restrictions on both *Pell's Equation* and *Continued Fraction*, and they will be introduced to distinctive solutions to different *Pell's Equation*.

1. Introduction

Pell's Equation has been one of the most important and interesting equations in the *Number Theory*. Its history is as interesting as the equation itself. Although it is believed that *Pell's Equation* was first studied by John Pell in the seventeenth century, the history of *Pell's Equation* dates back to times of Indian mathematician Brahmagupta and Greek mathematician Pythagoras [4]. Later, William Brouncker became the first European to solve the equation, and Leonhard Euler mistakenly attributed William Brouncker's solution to John Pell, which explains why the equation is named after John Pell [4].

Pell's Equation has other forms, *Generalized Pell's Equation* and *Negative Pell's Equation*. Although the *Generalized Pell's Equation* is not used as commonly as the *Pell's Equation*, we still provided definition, solutions, and explanations for it. However, the *Negative Pell's Equation* is still subject to various different research and experiments. Thus, finding absolute information on it is still relatively hard. Owing to this, we did not talk about *Negative Pell's Equation* in this paper.

Pell's Equation can be solved through many different ways, such as through *convergents* and *fundamental solution* via *Continued Fractions* or even through *Quantum Algorithms*. We can solve it by using trial-and-error method, or we can benefit from different theorems. In this paper, we mostly focused on solving it through *convergents*, *fundamental solutions*, and *Continued Fractions*.

Continued Fraction is highly essential for mathematicians and understanding the mysteries behind some special irrational numbers such as π, e, or ϕ. Furthermore, if we need to make calculations with the help of other irrational numbers such as $\sqrt{2}$, we take advantage of *Continued Fractions* without even realizing.

Since we were aware of the fact that we needed to introduce the *Pell's Equation*, its other

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types, and its different solutions, we gave detailed and articulable explanations and definitions.

Moreover, understanding the basics of the *Continued Fraction* is quite crucial to grasping the concept of the *Pell's Equation*. Due to this, we provided more than the basics of *Continued Fraction*. Thus, the section *Continued Fraction* can be used for other purposes, such as finding the best approximations to certain values, separately from understanding and solving the *Pell's Equation*.

In this paper, all the solutions and explanations were solved by us and calculators, and they were explained, defined and written by us. The accuracy of the solutions and explanations were checked through OEIS, research papers from distinguished researchers, and calculators.

2. Pell's Equation

2.1. **Pell's Equation.** *Pell's Equation* is a *Diophantine equation*. *Pell's Equation* are any equations where x and y are integers, and d is a positive integer but not a perfect square. That is,

$$x^2 - dy^2 = 1$$

The equation is extremely important in *Number Theory* since it comes with investigation and solution of numbers that are figurate in more than one way [6]. *Pell's Equation* can give infinite number of solutions.

Definition 2.1. *Diophantine equations* are polynomial equations involving only sums, powers, and products. All the constants are integers, and the only solutions of interest are integers. That is,

$$x^2 - y^2 = z^2$$

where x, y, and z are integers.

Proposition 2.2. *The reason why d cannot be a perfect square is that when d becomes a perfect square, we can only get one fundamental solution that is $(\pm 1, 0)$ for any positive integer d.*

Proof. Let d be 4, a perfect square. Then, we have

$$x^2 - 4y^2 = 1$$

$$x^2 - (2y)^2 = 1$$

The only perfect squares that are 1 apart are $|1|$ and 0. Thus, the only solution is $(\pm 1, 0)$. ■

Proposition 2.3. *The reason why d cannot be a negative integer is that when d becomes a negative integer, we cannot get infinite number of solutions.*

Proof. Let d be -1, a negative integer. Then, we have

$$x^2 - (-1y^2) = 1$$

$$x^2 + y^2 = 1$$

Thus, the only solutions to this equation are $(\pm 1, 0)$ and $(0, \pm 1)$.

■

Proof. Let d be any negative integer that is smaller than -1. Thus, we will let d be -2. Then, we have

$$x^2 - (-2y^2) = 1$$
$$x^2 + 2y^2 = 1$$

Thus, the only solution to this equation is $(\pm 1, 0)$. That is, y cannot be greater or smaller than 0 because if it becomes any integer other than 0, then both x^2 and $-dy^2$ will become greater than 1, which cannot happen.

■

Definition 2.4. *Fundamental solution* refers to any solution which can solve one or more *root causes*. Thus, the root of the problem is used to construct theorems and problems based on them. That is, the *fundamental solution* of an equation is the smallest solution to that equation.

2.2. **Generalized Pell's Equation.** *Generalized Pell's Equation* is the equation where x and y are integers and d is any positive integer which is not a perfect square, and the solution is any integer except 1. That is,

$$x^2 - dy^2 = n$$

Generalized Pell's Equation uses its *fundamental solution* and its *Pell's Equation* form's *fundamental solution* to provide other solutions. We will talk about this in more detail later in this paper.

3. Continued Fraction

3.1. **Continued Fraction.** *Continued Fraction* is any fraction whose numerator is an integer and denominator is a quantity plus a fraction, and the fraction's numerator and denominator follow a similar pattern. That is,

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{\ddots}}}}$$

or

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots + \cfrac{1}{a_{n-1} + \cfrac{1}{a_n}}}}}$$

depending on whether they are *Finite Continued Fraction* or *Infinite Continued Fraction.*

Moreover, they can be represented in *abbreviated notation* as

$$[a_0; a_1, a_2, a_3, \ldots]$$

or

$$[a_0; a_1, a_2, \ldots, a_{n-1}, a_n]$$

depending on whether they are *Finite Continued Fraction* or *Infinite Continued Fraction*. We will discuss about them later in this paper.

a_0 is called the *integer* part, whereas other values are called *partial numerator* and *partial denominator*. Any $\mathbb{R}$ can be represented by a continued fraction. We will prove this in this paper later in the subsections *Finite Continued Fraction* and *Infinite Continued Fraction*.

In most of the cases, when mathematicians use the *Continued Fraction* term, they use it to refer to *Regular Continued Fraction*. Thus, in this paper, when we use the *Continued Fraction* term, we will use it to refer to *Regular Continued Fraction* which we will talk in more detail later in this paper.

Definition 3.1. *Convergent* is what we get when we truncate a continued fraction after some number of terms. In *Continued Fraction*, *convergent* is the best approximation of that number. In other words, let $[a_0; a_1, a_2]$ be *Continued Fractions* of a $\mathbb{R}$. Then, $[a_0]$, $[a_0; a_1]$, and $[a_0; a_1, a_2]$ will be the *convergents* of that number. Respectively, they are denoted as $[C_0; C_1, C_2]$.

For example, the *convergents* of

$$\frac{93}{17} = 5 + \frac{1}{2 + \frac{1}{8}}$$

are [5], [5;2], and [5;2,8].

Moreover, $\sqrt{2}$ can be easily calculated by using the formula

$$\frac{numerator + (2.denominator)}{numerator + denominator}$$

Proof. The first 4 *convergents* of $\sqrt{2}$ are

$$\frac{1}{1}, \frac{3}{2}, \frac{7}{5}, \frac{17}{12}$$

To find the *convergents* of $\sqrt{2}$ through this formula, first, we need to find the place of $\sqrt{2}$ on the numerical axis. $\sqrt{2}$ is greater than 1 and smaller than 2. Thus, its *Continued Fraction* expression is

$$\sqrt{2} = 1 + \frac{1}{\ddots}$$

Due to this, the first *convergent* is (1,1). Since now we have the first *convergent*, we will start applying the formula.

$$\frac{1 + (2.1)}{1 + 1} = \frac{3}{2}$$

Now we have the second *convergent* (3,2), we will continue applying the formula.

$$\frac{3 + (2.2]}{3 + 2} = \frac{7}{5}$$

We will apply the formula once more with the third *convergent* (7,5).

$$\frac{7 + (2.5)}{7 + 5} = \frac{17}{12}$$

The fourth *convergent* is (17,12). Moreover, these solutions prove that this formula can be used to find the *convergents* of $\sqrt{2}$.

■

3.2. **Finite Continued Fraction.** *Continued Fraction* can either terminate at some point or go up endlessly. If the former happens, it will be called as *Finite Continued Fraction.* Where $n > 0$, it can be represented as either

$$c_0 = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots + \cfrac{1}{a_{n-1} + \cfrac{1}{a_n}}}}}$$

or

$$[a_0; a_1, a_2, \ldots, a_{n-1}, a_n]$$

Example 3.2. *The Continued Fraction of* $\frac{149}{17}$ *is*

$$149 = \boxed{8}.17 + 13$$

$$17 = \boxed{1}.13 + 4$$

$$13 = \boxed{3}.4 + 1$$

$$4 = \boxed{4}.1 + 0$$

Thus, it is

$$\frac{149}{17} = 8 + \cfrac{1}{1 + \cfrac{1}{3 + \cfrac{1}{4}}}$$

or

$$[8; 1, 3, 4]$$

Proposition 3.3. *If the first coefficient is an integer, and other coefficients are positive integers, then every rational number can be represented as a Finite Continued Fraction, and every Finite Continued Fraction represents a rational number.*

Example 3.4. *The Continued Fraction of* $\frac{119}{18}$ *is*

$$\frac{119}{18} = 6 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{3}}}}}$$

or

$$[6; 1, 1, 1, 1, 3]$$

Example 3.5. *The rational representation of* $2 + \cfrac{1}{1 + \cfrac{1}{13}}$ *is*

$$\frac{1}{13} + \frac{13}{13} = \frac{14}{13}$$

$$\frac{1}{\frac{14}{13}} = \frac{13}{14}$$

$$\frac{28}{14} + \frac{13}{14} = \frac{41}{14}$$

3.3. **Infinite Continued Fraction.** In the *Finite Continued Fraction* subsection, we mentioned about the way that the *Continued Fractions* either terminate at some point or go up endlessly. We already talked about the former one representing the *Finite Continued Fractions*. Thus, we will talk about the latter one, *Infinite Continued Fractions*, now.

Infinite Continued Fraction is the type of *Continued Fraction* where the fraction never stops. Where $n > 0$, it can be represented as either

$$c_0 = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{a_4 + \cfrac{1}{\ddots}}}}}$$

or

$$[a_0; a_1, a_2, a_3, a_4, \ldots]$$

Example 3.6. *The Continued Fraction of* ϕ *(Golden Ratio) is [3]*

$$\frac{1+\sqrt{5}}{2} = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{\ddots}}}}}$$

or

$$[1; 1, 1, 1, 1, \ldots]$$

Proposition 3.7. *If the first coefficient is an integer, and other coefficients are positive integers, every irrational number can be represented as an Infinite Continued Fraction, and every Infinite Continued Fraction represents an irrational number.*

Example 3.8. *One of the Continued Fractions of* π *is*

$$\pi = 3 + \cfrac{1}{7 + \cfrac{1}{15 + \cfrac{1}{1 + \cfrac{1}{292 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{3 + \cfrac{1}{1 + \ddots}}}}}}}}}}}$$

Example 3.9. *The irrational representation of* $1 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \ddots}}}}}$ *is* $\sqrt{2}$*. Because*

$$\sqrt{2} = 1 + (\sqrt{2} - 1)$$
$$\frac{\sqrt{2} - 1}{1} . \frac{\sqrt{2} + 1}{\sqrt{2} + 1} = \frac{2 - 1}{\sqrt{2} + 1} = \frac{1}{1 + \sqrt{2}}$$
$$1 + (\sqrt{2} - 1) = 1 + \frac{1}{1 + \sqrt{2}}$$
$$\sqrt{2} = 1 + (\sqrt{2} - 1) = 1 + \cfrac{1}{1 + (1 + \cfrac{1}{1 + \sqrt{2}})}$$
$$1 + \cfrac{1}{1 + (1 + \cfrac{1}{1 + \sqrt{2}})} = 1 + \cfrac{1}{2 + \cfrac{1}{1 + \sqrt{2}}}$$
$$1 + \cfrac{1}{2 + \cfrac{1}{1 + \sqrt{2}}} = 1 + \cfrac{1}{2 + \cfrac{1}{1 + (1 + \cfrac{1}{1 + \sqrt{2}})}} = 1 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{1 + \sqrt{2}}}}$$

3.4. **Regular Continued Fraction.** Mostly being referred to as just the *Continued Fraction*, *Regular Continued Fraction* is where the *partial numerators* are equal to 1. That is, $b_n = 1$ for all $n = 1, 2, \ldots$ It is an expression of the

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{\ddots}}}}$$

or

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots + \cfrac{1}{a_{n-1} + \cfrac{1}{a_n}}}}}$$

and

$$[a_0; a_1, a_2, a_3, \ldots]$$

or

$$[a_0; a_1, a_2, \ldots, a_{n-1}, a_n]$$

Example 3.10. *Let a_0 be $\frac{53}{7}$, a rational number. Then, we will have*

$$\frac{53}{7} = 7 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{3}}}$$

or

$$[7; 1, 1, 3]$$

Example 3.11. *Let a_0 be e, an irrational number. Then, we will have*

$$e = 2 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{4 + \cfrac{1}{1 + \cfrac{1}{1 + \ddots}}}}}}}$$

or

$$[2; 1, 2, 1, 1, 4, 1, 1, \ldots]$$

Regular Continued Fraction is accepted as one of the best methods to find near *commensurability*. *Commensurability* means *having a common ground* generally. In mathematics, *commensurability* is found when the ratio of two non-zero $\mathbb{R}$ is a rational number. Sometimes, *finding commensurability by using Regular Continued Fractions* is called as *Commensurable Continued Fractions.*

Example 3.12. *The Metonic Cycle used by Greeks to use to calculate time and date had 235 Lunar months that were nearly equal to 19 Solar years [2]. When we use Regular Continued Fraction, we will get*

$$\frac{235}{19} = 12\cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{2}}}}$$

and

$$[12; 2, 1, 2, 2]$$

The result we get is the sixth convergent of the ratio of the lunar phase(synodic) period and solar period.

Note 3.13. However, while reading the *Regular Continued Fraction*, we need to bear in the mind that mathematicians prefer different ways to notate the *Regular Continued Fraction*. Some of them use *comma*, instead of *semicolon* to write in abbreviated notation form, $[a_0, a_1, a_2, \ldots]$ instead of $[a_0; a_1, a_2, \ldots]$ Moreover, some of them notate as $[b_0; b_1, b_2, \ldots]$ instead of $[a_0; a_1, a_2, \ldots]$ Lastly, some of them notate by skipping the a_0, $[a_1; a_2, a_3, \ldots]$ instead of $[a_0; a_1, a_2, \ldots]$ For the sake of this paper, we will use

$$[a_0; a_1, a_2, \ldots]$$

or

$$[a_0; a_1, a_2, \ldots, a_n]$$

3.5. **Generalized Continued Fraction.** *Generalized Continued Fraction* is the generalization of *Regular Continued Fraction*. It is an expression of the form

$$a_0 + \cfrac{b_1}{a_1 + \cfrac{b_2}{a_2 + \cfrac{b_3}{a_3 + \cfrac{b_4}{\ddots}}}}$$

where *the partial numerators (b_1, b_2, ...)* and *partial denominators (a_0, a_1, ...)* can be complex numbers, integers, functions, or real numbers. That is, they are *arbitrary* values. Moreover, *Generalized Continued Fraction* has different forms such as

$$x = b_0 + \frac{b_1}{a_1} + \frac{b_2}{a_2} + \cdots$$

Generalized Continued Fraction is pretty useful to calculate irrational numbers, especially π, and thus provides couple of different generalizations of *Continued Fraction* of π. We will now introduce the 3 best-known *Generalized Continued Fractions* of π:

1. The *Leibniz Formula* for the *Generalized Continued Fraction* of π [7]:

$$\pi = \cfrac{4}{1 + \cfrac{1^2}{2 + \cfrac{3^2}{2 + \cfrac{5^2}{2 + \ddots}}}}$$

2. The *Nilakantha Somayaji's Formula* for the *Generalized Continued Fraction* of π [1]:

$$\pi = 3 + \cfrac{1^2}{6 + \cfrac{3^2}{6 + \cfrac{5^2}{6 + \cfrac{7^2}{6 + \ddots}}}}$$

3. The formula derived from *William Brouncker's formula* for the *Generalized Continued Fraction* of π [5]:

$$\pi = \cfrac{4}{1 + \cfrac{1^2}{3 + \cfrac{2^2}{5 + \cfrac{3^2}{7 + \ddots}}}}$$

Generally, it is assumed that the numerator of all fractions is 1. If the numerator or denominator become any other values such as *arbitrary values*, then we will have *Generalized Continued Fraction.*

Definition 3.14. *Arbitrary* means *not assigned to a specific value* in mathematics. That is, we could say

$$x + x = 2x$$

is true for *arbitrary* numbers for x=$\mathbb{R}$. However, we could not say

$$x + x = 2$$

is true for *arbitrary* numbers of x since x has a specific value which is 1.

Or, in other words, *arbitrary* means *all* in mathematics. Saying

"*For all a,b,* $a + b = b + a$"

is same as saying

"*For arbitrary a,b,* $a + b = b + a$"

4. Solutions to Pell's Equation

4.1. **Through Fundamental Solutions and Convergents.** As aforementioned in the subsection *Definition 2.4*, *fundamental solutions* are the root of the problems that are used to construct theorems and problems based on them. They are essential to find solutions to *Pell's Equation.* Finding the *fundamental solution* should be the priority as it is a lot easier to find other solutions based on the *fundamental solution.* Throughout this topic, we will only focus on the positive integers x and y to give more understandable solutions and explanations.

The easiest and the most basic method to find the *fundamental solution* is through trial-and-error method. If d is a small number, this method can come as handy.

Theorem 4.1. *By changing the value of y, then basing the value of x on it, we can find the fundamental solutions to the most basic Pell's Equations.*

Proof. Let d be 2. Then, we will have

$$x^2 - 2y^2 = 1$$

We can start by making y equal to 1. Then, we will have

$$x^2 - (2.1^2) = 1$$
$$x^2 - 2 = 1$$
$$x^2 = 2 + 1 = 3$$
$$x = \sqrt{3}$$

However, x should be an integer; thus, y cannot be 1.

Then, we will make y equal to 2. Then, we will have

$$x^2 - (2.2^2) = 1$$
$$x^2 - 8 = 1$$
$$x^2 = 9$$
$$x = 3$$

Since both x and y are integers, (3,2) is the *fundamental solution* to the *Pell's Equation* in which d is 2.

■

However, this method does not come in handy when we encounter greater values of d. Due to this, we use *Continued Fractions*, specifically *convergents*, for the greater values of d.

Theorem 4.2. *The fundamental solution of a Pell's Equation can be found by finding and and testing each consecutive convergents of $\sqrt{d}$ until a solution is found.*

Proof. Let d be 6, a greater value than 2. Then, we will have

$$x^2 - 6y^2 = 1$$
$$\alpha = x - \sqrt{6}y = 1$$

First, we need to find the *convergents* of $\sqrt{6}$. Thus, in order to that, we must find its place on the numerical axis. It is greater than 2 and smaller than 3. Thus, it can be written as

$$\sqrt{6} = 2 + z$$
$$(\sqrt{6})^2 = (2 + z)^2$$
$$6 = 4 + 4z + z^2$$
$$2 = z(4 + z)$$

Now, we need to find z.

$$z = \frac{2}{4 + z}$$
$$z = \frac{1}{2 + \frac{z}{2}}$$

This means that

$$\sqrt{6} = 2 + \cfrac{1}{2 + \cfrac{z}{2}}$$

And, $\frac{z}{2}$ is

$$\frac{z}{2} = \frac{1}{4+z}$$

Thus, now we have

$$\sqrt{6} = 2 + \cfrac{1}{2 + \cfrac{1}{4+z}}$$

However, we know the value of z. Thus, we now have

$$\sqrt{6} = 2 + \cfrac{1}{2 + \cfrac{1}{4 + \cfrac{1}{2 + \cfrac{z}{2}}}}$$

As you may have noticed, this expression goes on forever. Therefore, the *Continued Fraction* expression of $\sqrt{6}$ in *abbreviated notation* is

$$[2; \overline{2, 4}]$$

Since we now have the *Continued Fraction* expression of $\sqrt{6}$, we can start finding the *convergents* of $\sqrt{6}$. In order to find *convergents*, we can either apply a formula, similar to the formula we introduced in subsection *Definition 3.1*, or we can make z equal to 0 at any given stage. For $\sqrt{6}$, we will let z be equal to 0 at any given stage.

The first stage is $\sqrt{6} = 2 + z$; therefore, our first *convergent* is 2.

The second stage is $\sqrt{6} = 2 + \frac{1}{2+z}$; therefore, our second *convergent* is $\frac{5}{2}$.

The third stage is $\sqrt{6} = 2 + \cfrac{1}{2 + \cfrac{1}{4+z}}$; therefore, our third *convergent* is $\frac{22}{9}$.

The fourth stage is $\sqrt{6} = 2 + \cfrac{1}{2 + \cfrac{1}{4 + \cfrac{1}{2+z}}}$; therefore, our fourth *convergent* is $\frac{49}{20}$.

Since $\sqrt{6}$ is an irrational number, it is expressed in the *Infinite Continued Fraction* form, and it has infinite numbers of *convergents*. Owing to this, we will only focus on the first 4 *convergents*. They are

$$\frac{2}{1}, \frac{5}{2}, \frac{22}{9}, \frac{49}{20}$$

Now, we need to find the *fundamental solution* by applying these *convergents* to our *Pell's Equation* in which d equals to 6.

$$x^2 - 6y^2 = 1$$

We will start with the first *convergent*, (2,1). Now, we have

$$2^2 - 6.1^2 = -2$$

Thus, the first *convergent*, (2,1) is not the *fundamental solution*. So, we will apply the second *convergent*. Now, we have

$$5^2 - 6.2^2 = 1$$

Thus, the second *convergent* is the *fundamental solution*. We do not need to apply other successive *convergents* to find the *fundamental solution* anymore. However, they will come in handy when we try to find other solutions to *Pell's Equation*.

■

4.2. **Other Solutions of Pell's Equation.** In the previous subsection, we introduced how to find the *fundamental solutions* to each *Pell's Equation* through trial-and-error method and *Continued Fractions*, specifically *convergents*. Those 2 methods still could be used to determine the other solutions to *Pell's Equations*. Especially, using *Continued Fractions* is still one of the most used methods. Owing to this, we will start looking for other solutions, apart from the *fundamental solution*, using the aforementioned method.

Theorem 4.3. *By finding the convergents of $\sqrt{d}$ through Continued Fractions and applying the convergents to Pell's Equation, other solutions to the Pell's Equation can also be found.*

Proof. Since we have already found the first 4 *convergents* and the *fundamental solution* of $\sqrt{6}$, we will try to find the next solution by applying the other 2 *convergents* to our equation. Our equation is

$$x^2 - 6y^2 = 1$$

And, the other 2 *convergents* are

$$\frac{22}{9}, \frac{49}{20}$$

Now, we will start applying the third *convergent*, (22,9). Thus, we will have

$$22^2 - 6.9^2 = -2$$

Since the equation is not equal to 1, the third *convergent*, (22,9), is not the second solution to our equation. Due to this, we will apply the fourth *convergent*, (49,20). Thus, we will have

$$49^2 - 6.20^2 = 1$$

Since the equation is equal to 1, the fourth *convergent* is the second solution to *Pell's Equation* in which d equals to 6.

■

Although this method is relatively easier and less time-consuming than the trial-and-error method, still, this method also does not come in handy when we try to find other solutions, apart from the first and second solution, to *Pell's Equation*. Moreover, due to this, we will introduce another theorem and method.

Theorem 4.4. *By using the equation $(x + y\sqrt{d})^n = \alpha^n$, raising n to different powers, and then applying the fundamental solution, we can find the other solutions to Pell's Equation.*

Proof. Since it is the smallest value that d can get, we will make d equal to 2. Thus, we will use

$$x^2 - 2y^2 = 1$$

Now, we will use the equation which we introduced in the *Theorem 4.4*. Thus, we will have

$$x + \sqrt{2}y = \alpha$$

Since, now we have the equation, we need to start raising its powers. Thus, we will have

$$(x + \sqrt{2}y)^n = \alpha^n$$

Now, we need the *fundamental solution* to apply to equation above. In the *proof* part of the subsection *Theorem 4.1*, we have already found it through trial-and-error method. It is (3,2). Thus, we will now have

$$(3 + 2\sqrt{2})^n = \alpha^n$$

We know that when $n = 1$, the solution is (3,2). Due to this, we will start by making n equal to 2. Thus, we will have

$$(3 + 2\sqrt{2})^2 = \alpha^2$$

$$9 + 12\sqrt{2} + 8 = 17 + 12\sqrt{2} = \alpha$$

Now, we need to test whether this solution, (17,12), satisfies the *Pell's Equation* or not.

$$17^2 - 2.12^2 = 1$$

Therefore, our second solution to the *Pell's Equation* in which d equals to 2 is (17,12).

In order to find the third solution, we will make n equal to 3. Thus, we will have

$$(3 + 2\sqrt{2})^3 = \alpha^3$$

$$27 + 54\sqrt{2} + 72 + 16\sqrt{2} = 99 + 70\sqrt{2}$$

Now, we need to test whether this solution, (99,70), also satisfies the *Pell's Equation* or not.

$$99^2 - 2.70^2 = 1$$

Therefore, our third solution to the *Pell's Equation* in which d equals to 2 is indeed (99,70). ■

This method is one of the easiest and fastest ways to find the solutions to *Pell's Equation*. This method, particularly, comes in handy when we encounter greater values than 2. Moreover, this method can be used to find solutions to *Generalized Pell's Equation* as well.

4.3. **Solutions to Generalized Pell's Equation.** Finding solutions to *Generalized Pell's Equation* is relatively harder and more time-consuming than finding solutions to *Pell's Equation*. As usual, trial-and-error method can be used to find the solutions. However, apart from finding the *fundamental solution*, it does not come in handy. In order to spend less time on finding the solutions to *Generalized Pell's Equation*, we take help from *Pell's Equation*.

Theorem 4.5. *To find the other solutions to Generalized Pell's Equation, we first need to find its fundamental solution. Then, we need to make n=1 and find that equation's fundamental solution. Then, we need to apply their fundamental solutions to the equation* $(x + y\sqrt{d})^n = \alpha^n$ *and raise their powers. Lastly, we need to multiply both of them.*

Proof. First, we will start by selecting a *Generalized Pell's Equation* to make calculations to it. In this *proof* section, we will only focus on the positive integer x and y. Since it takes less time, we will use

$$x^2 - 6y^2 = 3$$

First, we need to find its *fundamental solution*. We will do it through trial-and-error method, and we will start with making y equal to 1. Thus, we will have

$$x^2 - 6y^2 = 3$$
$$x^2 = 6y^2 + 3$$
$$x^2 = 6.1^2 + 3 = 9$$
$$x = 3$$

Therefore, the *fundamental solution* to our *Generalized Pell's Equation* is (3,1). Now, we will apply it to the equation we gave in the *Theorem 4.5*. We will leave the power at 1. Thus, we will have

$$3 + 1\sqrt{6} = \alpha$$

Now, we will make the equation equal to 1.

$$x^2 - 6y^2 = 1$$

We have already found the *fundamental solution* to the *Pell's Equation* in which d equals to 6 in the *proof* part of the subsection *Theorem 4.2*. It was (5,2). We will again leave the power at 1, and in order to avoid confusion, we will replace α with β. Thus, we will have

$$5 + 2\sqrt{6} = \beta$$

Now, we need to multiply them. In order to find the second solution our *Generalized Pell's Equation*, we raised the n to 1. Thus, we will have

$$\alpha.\beta = (3 + 1\sqrt{6}).(5 + 2\sqrt{6})$$
$$\alpha.\beta = 15 + 5\sqrt{6} + 6\sqrt{6} + 12 = 27 + 11\sqrt{6}$$

Now, we need to test whether this solution, (27,11), satisfies our *Generalized Pell's Equation* or not.

$$27^2 - 6.11^2 = 3$$

Therefore, this solution, (27,11), is the second solution to our *Generalized Pell's Equation*, right after its *fundamental solution*. Moreover, in order to find third or other solutions to our *Generalized Pell's Equation* we can raise the power to 2 or other greater values. ■

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