Robinson–Schensted–(Knuth) correspondence

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Remark: Representation Theory

Young Diagrams are intimately connected with the represntation theory of symmetric groups. The formula mentioned in a couple of slides is the dimension identity of irreducible represntations.
Representation theory and Young Diagram have further applications to the rest of abstract algebra, and to physics, but we won't discuss this for the purposes of the talk.

Partitions

Definition 1

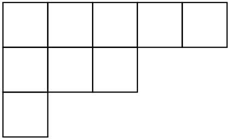
Let $n \in \mathbb{Z}_{\geq 0}$. Define a **partition** of n to be a weakly decreasing set of integers $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r > 0$, such that $\sum_{i=1}^{r} \lambda_i = n$.
When λ is a partition of n, we denote $|\lambda| = n$, and call n the **size** of λ.
The **partition function** $p(n)$ is the number of partitions of n.

Young Diagrams

Definition 2

A **Young Diagram** of a partition λ is an array of left-justified boxes with the number of boxes in each row corresponding to the parts of the partition.

For example, when $\lambda = (5, 3, 1)$, we have the following Young diagram for λ:



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Young Tableaux

Definition 3

A **Young Tableau** is a filling of a Young diagram with *alphabets*, usually such alphabets have a *total order*, or are just positive integers.
A **Standard Young Tableau** (SYT for short), is a tableau such that the entries in each row and each column are increasing from left to right and from up to down. We usually use the numbers from 1 to n in a SYT.
We denote f^λ as the number of SYT's for the Young diagram with *shape* λ

Examples of Young Tableaux

This is an example of a standard tableau:

1	3	5
2	4	6
7		

This is an example of a non-standard tableau as 7 is to the left of 3, and 5 is above 3.

1	4	5
2	7	3
6		

The Statement of the Correspondence

Theorem 4

There exists a bijection between ordered pairs of same-shaped SYT's over all partitions for n and n-permutations.
Formulaically:

$$(n!) = \sum_{|\lambda|=n} (f^{\lambda})^2$$

is true $\forall\, n \in \mathbb{N}$

Schensted correspondence

We provide a heuristic algorithm:

Algorithm 5

Let $a_1 a_2 \ldots a_n$ be a permutation. We initially have the tableau with the single cell containing a_1. We take the remaining numbers in the partition and create a tableau of size m, by performing the following algorithm inductively:

Initialize $x_1 = a_i$ (we do this $n-1$ times for $i = 2, 3, \ldots n$)

- *If x_i is bigger than all numbers in the ith row, attach x_i at the end of the row*
- *If $\exists y > x_i$ in the ith row, replace y with x_i, and let $y = x_{i+1}$*

Examples

Let the permutation be 3124. We first have:

3

Next, we replace 3 by 1, and bring 3 in the second row

1
3

Examples

The next two steps we will add 2 and 4 to the first row, hence we have

1	2
3	

and finally

1	2	4
3		

Insertion and Recording Tableaux

The algorithm mentioned creates the **insertion tableau**. To create the bijection, we also create the **recording tableau**. This is to record in what iteration does the cell get created. For 3124, the recording tableau is

1	3	4
2		

You can create an inverse bijection from two same-shaped tableau to a permutation, thus proving the theorem.

Another Example

Let's construct the tableau for the permutation 51342:

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5	

-

1	
5	

-

1	3
5	

Another Example

1	3	4
5		

1	2	4
3		
5		

The recording tableau being:

1	3	4
2		
5		

Knuth's extension

You can define a *plane partition* to be an ""Young Diagram-shaped" array of two-dimensional positive integers that are weakly decreasing from left to right and top to bottom. For example:

6	4	3	1
4	4		
2			

Knuth's extension

A *semistandard tableau* is tableau in which we allow the rows to weakly increase instead of strongly increase. For example,

1	1	3
2	2	
5		

There is a similar bijection from plane partitions to pairs of semistandard tableau.

Thanks!