YOUNG DIAGRAMS AND TABLEAUX

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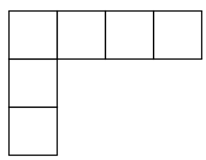
ABSTRACT. We first define Young Diagrams and Young Tableaux, and prove preliminary results about those structures. We then show the Robinson Schensted correspondence, and Knuth's generalisation. Next, we introduce representation theory and state its connection with Young Tableaux. Finally, we define graded posets of rank n, Sperner posets, posets quotiented by a group

1. Introduction

Partitions have been studied since antiquity. Traditionally, there have been two main ways to represent partitions pictorially, Ferrer diagrams and Young diagrams. This paper is concerned with the later representation. A Young diagram is an array of boxes in which the rows represent the sizes of the parts of the partitions. For example the partition $(4, 1, 1)$ has Young Diagram:



We can fill numbers in Young Diagram. These are called Young Tableaux. For the example above, we could have the following Young Tableaux:

1	3	5	6
2			
4			

The tableau above is also standard as each row and column have numbers strictly increasing from left to right and from up to down.
If n if a fixed natural number, We can use Young Diagrams and a conjugation argument to prove that the number of partitions of n with m parts equal the number of partitions of n with largest part m.

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One can use the Robinson-Schensted correspondence to give bijection from two same shaped SYTs (Standard Young Tableaux) of size n to all n permutations. We construct an algorithm that creates a tableau by "inserting' numbers from a permutation (represented by a string).
We then also construct a recording tableau, whose numbers represent when each cell in the insertion tableau were created. This gives the desired bijection we require.
We also provide a similar algorithm in the case of two-line arrays (a two dimensional generalisation of partitions).
A two dimensional array is a $2 \times m$ matrix with positive integers, with some sort of lexicographic order. For example:

$$\begin{pmatrix} 1 & 1 & 1 & 3 & 3 & 4 & 5 \\ 1 & 1 & 3 & 2 & 3 & 2 & 1 \end{pmatrix}$$

is a two dimensional array as the first row is weakly increasing, and the second row is also weakly increasing provided the numbers they correspond to in the first row are the same (this is similar to the usual lexicographic order on the complex numbers $\mathbb{C}$.

Representations are homomorphisms from a group to the group of all invertible operators on some vector space V. If the dimension of V is finite, then we can also create a group homomorphism for G to the group of all $n \times n$ invertible matrices in the base field of V.
The notion of irreducible representations is crucial, as it helps us decompose any representation as a direct sum of irreducible representations (at least when the group related to the representation is finite).
The number of isomorphic classes of irreducible representations of G can be described group theoretically: it's equal to the number of conjugacy classes of G.
The formula derived from the Robinson-Schensted correspondence is a special case of the formula describing the relation between $|G|$ and the dimensions of the irreducible representations.

Some partially ordered sets can be graded of a certain finite rank n. One example is the Boolean algebra B_n of all subsets of the n-element set $\{1, 2, \ldots, n\}$.
A Sperner poset is a partially ordered set where the largest antichain (a subset in which in two elements are comparable) has the same size of the largest rank set (we define ranks in a graded poset n in the body of the paper).
We state that B_n has the Sperner property, and so does B_n/G (the Boolean algebra quotiented by a group action G on B_n).
Lastly, we state that the poset $L(m, n)$ (of partitions with at most m parts and with largest part n) is isomorphic to B_{mn}/G_{mn}, where G_{mn} is the permutation group acting on a $m \times n$ rectangles with cells, in which two cells in the same row stay in the same row.

2. Definitions and the RSK Algorithm

We first go over a few tools that we would need throughout the paper.

Definition 2.1. Let $n \in \mathbb{Z}_{\geq 0}$. Define a **partition** of n to be a weakly decreasing set of integers $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r > 0$, such that $\sum_{i=1}^{r} \lambda_i = n$.
When λ is a partition of n, we denote $\lambda \vdash n$ or $|\lambda| = n$, and call n the **size** of λ.
The **partition function** $p(n)$ is the number of partitions of n.

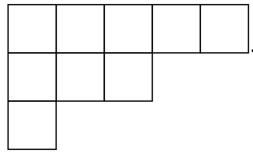
Rarely, we add an infinite string of zeros after the all non-zero terms of the partition, this is done to compare partitions of different sizes with the following order:

Definition 2.2. Let $\lambda = (\lambda_1, \lambda_2, \ldots)$ and $\mu = (\mu_1, \mu_2, \ldots)$ be partitions (perhaps of different sizes). We say $\lambda \geq \mu$ if $\lambda_i \geq \mu_i$ for all natural numbers i (this is commonly called a *lexicographic order*).
This gives rise to a partial order Y on the set of all partitions of all whole numbers (including $\emptyset = (0, 0, \ldots)$, and the poset (partially ordered set) formed is called **Young's lattice**.
The restriction of the poset Y on the set of partitions of with at most m non-zero parts and with largest part at most n is denoted by $L(m, n)$.

The poset defined above is called a *lattice* as it has additional properties that make it a lattice (which we shall not go over for the purposes of this paper).
We will now define the most important structure of the paper, namely the *Young Diagram* of a partition.

Definition 2.3. A **Young Diagram** of a partition λ is an array of left-justified boxes with the number of boxes in each row corresponding to the parts of the partition.

This is pretty vague, so it's important to show an example. When $\lambda = (5, 3, 1)$, we have the following Young diagram for λ:



Definition 2.4. A **Young Tableau** is a filling of a Young diagram with *alphabets*, usually such alphabets have a *total order*, or are just positive integers.
A **Standard Young Tableau** (SYT for short), is a tableau such that the entries in each row and each column are increasing from left to right and from up to down. We use the positive integers from 1 to n exactly once for this.

This is an example of a standard tableau:

1	3	5
2	4	6
7		

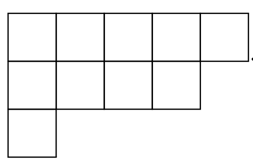
This is an example of a non-standard tableau as 7 is to the left of 3, and 5 is above 3.

1	4	5
2	7	3
6		

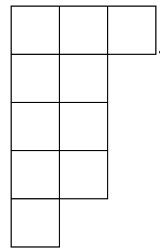
Let's prove a couple of basic theorems:

Proposition 2.5. *Let $n \in \mathbb{N}$ be fixed. The number of partitions of n with m parts is equal to the number of partitions of n with largest part m.*

Proof of Proposition: 2.5. Consider the Young diagram of a partition of n with m parts. For example, let's let $\lambda = (5, 4, 1)$, $n = 10$ and $m = 3$. So the Young diagram would be



For a partition λ, define the *conjugate partition* $\tilde{\lambda}$ to be the partition that is associated with the "reversed Young diagram". This means that the rows of λ's Young diagram become the columns of $\tilde{\lambda}$'s Young diagram, and vice versa (note that conjugation is an involution, meaning that conjugating a partition twice gives back the original partition). Hence, the Young diagram for $\tilde{\lambda}$ would look like



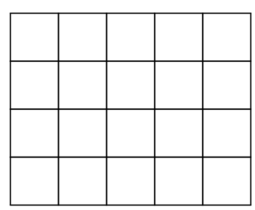
corresponding to the partition $(3, 2, 2, 2, 1)$. One can observe that if λ has m parts, then $\tilde{\lambda}$ has largest part m.
Now let A be the number of partitions of n with m parts, and B be the number of partitions with largest part m. The map $\sim_1$: $A \longrightarrow B$ (representing conjugation) is well defined according to the previous paragraph.
The above map has a two sided inverse, namely $\sim_2$: $B \longrightarrow A$ (also representing conjugation). Hence, $\sim_1$ is a bijection, and $|A| = |B|$, proving the theorem. ■

In particular, we have $L(m, n) = L(n, m)$. We know state and prove the cardinality of $L(m, n)$

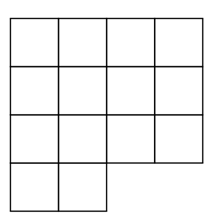
Proposition 2.6. $|L(m, n)| = \binom{m+n}{m}$

It easily follows from this proposition that we have $L(m, n) = L(n, m)$.

Proof of Proposition: 2.6. Any Young diagram of a partition λ belonging to $L(m, n)$ will fit into a $m \times n$ rectangle, perhaps like



To choose a specific Young diagram, we can construct a sequence of n "U's and m $R's$, corresponding to a lattice path starting from the bottom left corner, and ending with the top right corner, and only using up and right moves. For the rectangle shown above, if we have the sequence $RRURRUUR$, that would correspond to the Young Diagram



An U corresponds to moving to the next part with the same size, whereas a R corresponds to incrementing the size of the part on which the walk is at.
Since we have $m + n$ symbols to write, and m choices for the positions of the Rs, we hence we the count to equal $\binom{m+n}{m}$. ■

We can represent permutations (bijective functions σ on $\{1, 2, \ldots, n\}$) as a string of numbers from 1 to n (like $a_1a_2 \ldots a_n$ such that $a_i = \sigma(i)$.
Here is the statement of the Robinson-Schensted correspondence:

Theorem 2.7. *There exists a bijection between ordered pairs of same-shaped SYT's over all partitions for n and n-permutations.*
Formulaically,

$$n! = \sum_{\lambda \vdash n} (f^\lambda)^2$$

Proof of Theorem: 2.7. We'll prove this theorem algorithmically.

Algorithm 2.8. *[Robinson-Schensted correspondence] Let $a_1a_2 \ldots a_n$ be a permutation. We initially have the tableau with the single cell containing a_1. We take the remaining numbers in the partition and create a tableau of size m, by performing the following algorithm inductively: Initialize $x_1 = a_i$ (we do this $n-1$ times for $i = 2, 3, \ldots n$)*

- *If x_i is bigger than or equal to all numbers in the ith row, attach x_i at the end of the row*
- *Else if $\exists y > x_i$ in the ith row, replace the smallest and rightmost y (satisfying the inequality above) with x_i, and let $y = x_{i+1}$*

Here is an example of the algorithm:
Let the permutation be 3124. We first have:

$$\begin{array}{|c|} \hline 3 \\ \hline \end{array}$$

Next, we replace 3 by 1, and bring 3 in the second row

$$\begin{array}{|c|} \hline 1 \\ \hline 3 \\ \hline \end{array}$$

The next two steps we will add 2 and 4 to the first row, hence we have

$$\begin{array}{|c|c|} \hline 1 & 2 \\ \hline 3 \\ \cline{1-1} \end{array}$$

and finally

$$\begin{array}{|c|c|c|} \hline 1 & 2 & 4 \\ \hline 3 \\ \cline{1-1} \end{array}$$

The algorithm mentioned creates the **insertion tableau**. To create the bijection, we also create the **recording tableau**. This is to record in what iteration does the cell get created. For 3124, the recording tableau is

1	3	4
2		

We need to create the inverse bijection for the the theorem to be completely proven.

Suppose we have the following two Young Tableaux (the first being the insertion tableau; the second being the recording tableau)

1	2	4
3		

1	3	4
2		

The recording tableau shows that the top right cell was the last to be added. This means that, in the insertion tableau, the 4 was added to the first row last, hence the final number in our permutation must be 4. We then have the following tableaux

1	2
3	

1	3
2	

We can conclude that 2 was the second last number in the permutation. Continuing like this, we retrieve our original permutation 3124. ■

We now proceed to Knuth's generalisation.

Definition 2.9. A **two-line array** (occasionally called a *generalized permutation* is defined like

$$\begin{pmatrix} i_1 & i_2 & \dots & i_m \\ j_1 & j_2 & \dots & j_m \end{pmatrix}$$

where

- $i_1 \leq i_2 \leq \dots \leq i_m$
- If $i_u = i_v$ for some $u \leq v$, then $j_u \leq j_v$

We now define a structure similar to a SYT.

Definition 2.10. A **semistandard Young tableau** (SSYT for short) is a Young Tableau in which entries in the rows are weakly increasing from left to right, and entries in the columns are strictly increasing from up to down.

Hence, all SYT's are SSYTS (with the converse not being true).
Here's an example:

1	1	1	3	4
2	3	3		
5				

Similar to above, we can create a bijection from two-line arrays to pairs of same-shaped SSYTs. We'll give a brief sketch of the idea but not go too in depth.
We'll now try creating an algorithm similar to Algorithm 2.8. First, we use the insertion algorithm to create a tableau using the sequence $j_1 j_2 \dots j_m$.
For example, suppose we have the two-line array

$$\begin{pmatrix} 1 & 1 & 1 & 3 & 3 & 5 \\ 1 & 2 & 2 & 3 & 3 & 1 \end{pmatrix}$$

We then have the "j-sequence" to be 122331. Applying Algorithm 2.8 to the j-sequence, we get

1	1	2	3	3
2				

We can create a recording tableau, just like in the partition case:

1	2	3	4	5
6				

Finally, we places the entries $1, 2, 3, 4, 5$ and 6 with the "i-sequence", in this case $1, 1, 1, 3, 3$ and 5. This gives us another SSYT:

1	1	1	3	3
5				

We encourage the reader to go further in the study of the Robinson-Schensted-(Knuth) Algorithm.
For example, Viennot's geometric construction of the insertion and recording tableaux can help prove that, if a permutation σ is mapped to the pair of SYTs (P, Q), then the inverse of σ, σ^{-1} would be mapped to (Q, P).
Another fundamental fact regarding the algorithm is that the longest increasing subsequence of σ represented as a string is equal to the length of the first row of P (and Q).
Similarly, we have the fact that the longest decreasing subsequence of σ is equal to the length of the first column of P (and Q).
Both of these results are can be observed directly from the definition of the algorithm.

3. Group Representations

We'll now define basic concepts of *group representation*, in order to explain it's connection to Young Diagrams. We assume the reader is aware of the basic definitions of a group.

Definition 3.1. A **group representation** of a group G is a vector space V along with a group homomorphism $\phi : G \longrightarrow GL(V)$. Over here, $GL(V)$ is the space of all invertible linear operators on V, with group operation being function composition.
The **dimension** of a representation (V, ϕ) is the *vector space dimension* of V (provided that it is finite).

We usually assume that V is a finite dimensional complex vector space. In that case, we may note that there exists an isomorphism:

$$\psi : GL(V) \longrightarrow GL_n(\mathbb{C})$$

where n is the dimension of V and $GL_n(\mathbb{C})$ is the group of all invertible $n \times n$ matrices, with group operation being matrix multiplication. The isomorphism ψ essentially represents an operator $\mathcal{L} \in GL(V)$ as a matrix by making it act on a basis of V.
We can hence think of group representations as group actions being "represented" by matrices, by considering the homomorphism $\psi \circ \phi$ from G to $GL_n(\mathbb{C})$.

Definition 3.2. Let (V, ϕ_V) and (W, ϕ_W) be representations of G. A **homomorphism of representations** is a homomorphism of vector spaces $\Phi : V \longrightarrow W$, such that the following commutative diagram holds

$$\begin{array}{ccc} V & \xrightarrow{\Phi} & W \\ \big\downarrow{\scriptstyle \phi_V(g)} & & \big\downarrow{\scriptstyle \phi_W(g)} \\ V & \xrightarrow{\Phi} & W \end{array}$$

for every $g \in G$. Formulaically, this means that $\Phi \circ \phi_V(g) = \phi_W(g) \circ \Phi$ is true for every $g \in G$.
An **isomorphism of representations** is a homomorphism of representations where the function Φ is an isomorphism of vector spaces.

Naturally, we can define *equivalency* between representations, which leads to *isomorphism classes* of representations.

Definition 3.3. Let (V, ϕ) be a representation of G.
An **invariant subspace** is a subspace $W \subset V$ such that

- W is non-trivial ($W \neq 0$ or V)
- $\forall\, w \in W$ and $\forall\, g \in G$, we have $\phi(g)(w) \in W$

A **sub-representation** is a pair $(W, \phi|_W)$, where W is an invariant subspace of (V, ϕ) and $\phi|_W$ is the restriction of ϕ onto $GL(W)$

One can show that a sub-representation is well-defined (up to isomorphism) and satisfies the condition on a representation.

Definition 3.4. An **irreducible representation** is a representation (V, ϕ) that has no sub-representations (or invariant subspaces).

Recall the definition of the direct sum of two vector spaces V and W. An element of $V \bigoplus W$ is of the form $v + w$, where v and w are elements of V and W respectively. If $\mathcal{B}_V$ is a basis for V, and $\mathcal{B}_W$ is a basis for W, then it's easy to see that the set $\mathcal{B}_V \cup \mathcal{B}_W$ is a basis for $V \bigoplus W$.

Definition 3.5. If (V, ϕ_V) and (W, ϕ_W) are representations of G, then define the **direct sum representation** to be $(V \bigoplus W, \phi)$, where $\phi(g)(v + w) = \phi_V(g)(v) + \phi_W(g)(w)$

One can check that this definition is well defined.

Theorem 3.6 (Maschke's Theorem)**.** *Any representation (V, ϕ) of a finite group G is a direct sum of irreducible representations (of G).*

Irreducible representations are the "building blocks" of all representations, hence making them important to study and classify.
Recall that two elements $g_1, g_2 \in G$ are *conjugate* if $\exists\, h \in G$ such that $g_2 = h^{-1} g_1 h$. This can be proven to be an equivalence relation, and give rise to the *conjugacy classes* of G.

Theorem 3.7 (Corollary of Orthogonality Relations)**.** *Assume G is finite. There are finitely many isomorphism classes of irreducible representations for a given representation (V, ϕ) of G, the number of classes being equal to the number of conjugacy classes of G.*
If $d_1, d_2, \ldots d_k$ represent the dimensions of the irreducible representations of G, then d_i divides $|G|$ for all i, and

$$|G| = \sum_{i=1}^{k} d_i^2$$

Notice that the summation above is similar to the summation for the Robinson-Schensted correspondence:

$$n! = \sum_{\lambda \vdash n} (f^\lambda)^2$$

They are in fact the same equation, the latter summation is the general summation when applied to $G = S_n$, the symmetric group of all n-permutations, with group operation being function composition.
It is true that the number of conjugacy classes in S_n is equal to the number of partitions of n, this is because two permutations σ_1 and σ_2 are in the same conjugacy class if and only if they have the same number of cycle type (cycle type here being the size of the cyle).
For example, $(4,5)(1,2,3)(6)$ and $(6,4)(2,5,3)(1)$ are conjugate to each other.

4. Sperner property of $L(m, n)$

As mentioned before $L(m, n)$ is a poset with size $\binom{m+n}{m}$. Recall that a poset is a *relation* "$\leq$" on a set X such that

- $x \leq x$ for all $x \in X$
- $x \leq y$ and $y \leq x$ implies $x = y$
- $x \leq y$ and $y \leq z$ implies $x \leq z$

Definition 4.1. An element y **covers** x ins a poset X if $x < y$ and there exists no $z \in X$ such that $x < z < y$.

We can now define between maps between posets

Definition 4.2. A **poset homomorphism** between two posets $(X_1, \leq_1)$ and $(X_2, \leq_2)$ is a function $\psi : X_1 \longrightarrow X_2$ such that $x \leq y$ implies $\psi(x) \leq \psi(y)$
A **poset isomorphism** is a poset homomorphsim that is also a bijection.

If ψ is a poset isomorphism, then so is ψ^{-1}.
The fact that ψ^{-1} is bijective follows immediately from the fact that ψ is bijective.
When $u = v$, it's clear that $\psi^{-1}(u) \leq \psi^{-1}(v)$ (as $\psi^{-1}(u) = \psi^{-1}(v)$).
Assume $u < v$. For the purposes of contradiction, say we have $\psi^{-1}(u) > \psi^{-1}(v)$. Then, $\psi(\psi^{-1}(u)) > \psi(\psi^{-1}(v))$ (as ψ is a poset homomorphism).
This gives us $u > v$ (as $\psi \circ \psi^{-1} = id$). This contradicts $u < v$, hence proving the claim.
We now state more definitions related to posets:

Definition 4.3. A **totally ordered set** is a partially ordered set where every pair of elements are comparable. This means that at least one of $x \leq y$ or $y \leq x$ is true.
A **chain** is a totally ordered subset of a partially ordered set.

For example, if our poset is the Boolean algebra B_7 of all subsets of $\{1, 2, \ldots, 7\}$ (including the empty set $\emptyset$), then the following is an example of a chain in our poset:

$$\emptyset \subset \{1\} \subset \{1, 2\} \subset \{1, 2, 4, 5, 7\} \subset \{1, 2, 3, 4, 5, 6, 7\}$$

The above chain has *length* 4 as there are 4 subset signs ($\subset$s) in the set above.
We say that a finite poset is *graded of rank* n if every *maximal chain* in the poset has length n (maximal here meaning that it's not contained in any other chain).
All Boolean algebras B_n are graded of rank n, this is not hard to prove.
A chain $x_0 \leq x_1 \leq \ldots x_n$ of length n is said to be *saturated* if x_i covers x_{i-1} for every $i = 1, 2, \ldots, n$. A chain is *unsaturated* if it is not saturated (obviously).

Note that all maximal chains are saturated (because if a chain were unsaturated, it meant one could add an element in between two elements in the chain, meaning the chain is not maximal).
However the converse is not true, a chain can be saturated, meaning one couldn't add elements in between the chain, but one could still add element in the very beginning or the very end.
For example, if we have the following saturated chain in B_6

$$\{1, 2\} \subset \{1, 2, 6\} \subset \{1, 2, 3, 6\}$$

then, it's clear that every element covers the element before, but we can the element $\{1\}$ in

the beginning (or add the element $\{1, 2, 3, 5, 6\}$ in the end) to create a bigger chain contained in that chain, thus showing that the chain isn't maximal.

Definition 4.4. Let X be a finite poset that is graded of rank n. An element $x \in X$ has **rank** i if the largest saturated chain of X with largest element x has length X.
We denote $\rho(x) = i$ if x has rank i in the poset X.

Note that the singleton set x is a saturated chain, and with the fact that X is finite, the above definition is well-defined.
Note that the empty set $\emptyset$ has rank 0, and the full set X has rank n.

Definition 4.5. Let X_i be the set of all elements in X with rank i.
Let $x_i = |X_i|$ (the cardinality of X_i

It's clear then that $X = X_0 \sqcup X_1 \sqcup ... \sqcup X_n$ (where $\sqcup$ is the disjoint union operator.
For the set B_n, we have the sets $(B_n)_i$ like so:

$$(B_n)_i = \{x \subseteq \{1, 2, \dots, n\} : |x| = i\}$$

(Note that ":" means such that)

Since the number of element with cardinality i is $\binom{n}{i}$, we have that

$$(b_n)_i = \binom{n}{i}$$

A couple of more definitions:

Definition 4.6. A poset X that is graded of rank n is called **rank symmetric** if $x_i = x_{n-i}$ is true for all $i = 0, 1, 2, \dots, n$.

X is called **rank unimodal** if there exists $0 \leq i \leq n$ such that

$$x_0 \leq x_1 \leq \cdots \leq x_i \geq x_{i+1} \cdots \geq x_n$$

If we assume that our poset X (graded of rank n) is both rank symmetric and rank unimodal, then it's clear that

$$x_0 \leq x_1 \leq x_2 \cdots \leq x_{\frac{n}{2}} \geq \cdots \geq x_n$$

when n is even. When n is odd, we have

$$x_0 \leq x_1 \leq x_2 \cdots \leq x_{\lfloor \frac{n}{2} \rfloor} = x_{\lceil \frac{n}{2} \rceil} \geq \cdots \geq x_n$$

The Boolean algebra B_n is rank symmetric and rank unimodal.

It's a common fact that $|B_n| = 2^n$ (each element $1, 2, \ldots, n$ has two choices; either it is in an element of B_n, or it isn't).
We now define the "opposite" of a chain:

Definition 4.7. If X is a partially ordered set, then an **antichain** A contained in X is a set in which no two elements are comparable.
In other words, neither $x \leq y$ nor $y \leq x$ is true for any elements x and y in A

Every set X_i (as defined above) is an antichain in X.

This is because if there existed distinct x and y in X_i such that $x < y$, then we could create a chain larger than length i that has maximal element y (this can be done by adjoining y after x's i length chain where it's the maximal element).
We would like to count the size of the largest antichain in a poset.
Specifically for the Boolean algebra B_n. The size of the largest rank set (i.e, the $X_1, X_2, \ldots X_n$ list of sets with elements of same rank) in B_n is $\binom{m}{\lceil \frac{n}{2} \rceil}$ for all natural numbers n.

Definition 4.8. A poset X that is graded of rank n is called a **Sperner poset** (or has the *Sperner property*) if the largest antichain in X is that size of the largest rank set.
We can write this mathematically (let $A \triangleleft X$ denote that A is antichain inside of X):

$$\max\{|A| : A \triangleleft X\} = \max\{|X_i| : 0 \leq i \leq n\}$$

It turns out that B_n has the Sperner property.
This means that the largest antichain inside B_n has size $\binom{n}{\lceil \frac{n}{2} \rceil}$ (the size of the largest rank set (namely $X_{\lceil \frac{n}{2} \rceil}$)
This can be proven using linear algebra, however we won't go over that in this paper.
We now define Boolean algebras quotiented by finite groups

Definition 4.9. A **poset automorphism** on a poset X is a poset isomorphism from X to itself.
Let $\pi \in S_n$ be a permutation. π can **act** on a set $x \in B_n$ as follows:

$$\pi(x) = \pi(\{a_1, a_2, \dots, a_i\}) = \{a_{\pi(1)}, a_{\pi(2)}, \dots, a_{\pi(i)}\}$$

(the action above is actually known as a **group action**)

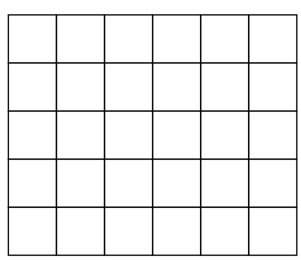
Let G be a subgroup of S_n. Define the **quotient poset** B_n/G to be the *orbits* of the group action of G on B_n
An orbit is the set of all elements that can be reached to one another via a group action. More specifically, an orbit is an equivalence class on the following equivalence relation:

$$x \sim y \iff \exists\ \pi \in G : \pi(x) = y$$

(this can be proven to satisfy the conditions on an equivalence relation).

Any group G is isomorphic to a subgroup of some S_n for some $n \in \mathbb{N}$ (this is known as Cayley's theorem).
It also turns out that B_n/G is also a graded of rank n, rank symmetric and unimodal, and Sperner. The proof will also be omitted, as we now try showing that $L(m, n)$ is isomorphic to B_n/G for some $n \in \mathbb{N}$ and for some group G.
We now construct this group G.

Definition 4.10. Say we have a $m \times n$ rectangle (call it R_{mn}).



For example, we have a 5×6 rectangle.

We have the group S_{mn} that permutes the cells of R_{mn}.
Define G_{mn}, a subgroup S_{mn} to be the group that contains permutations that

- permutes individual rows freely
- then permutes those rows itself

The definition above implies that $|G_{mn}| = m!(n!)^m$ (as there are m rows, in which we permute n cells, creating the term $(n!)^m$. The $m!$ comes from permuting the rows).
For example, if we have the following in R_{56}:

1	2	3	4	5	6
7	8	9	10	11	12
13	14	15	16	17	18
19	20	21	22	23	24
25	26	27	28	29	30

Then a valid permutation in G_{56} would be:

8	7	9	10	12	11
6	2	3	1	4	5
30	25	26	24	27	28
19	20	22	21	23	24
17	16	14	18	15	13

We omit the proof that $L(m, n)$ is isomorphic to B_{mn}/G_{mn} under poset isomorphism.
Note that G_{mn} is actually the *wreath product* of S_m and S_n
The fact that $L(m, n)$ has applications in number theory. Specifically, variants of the Erdös-Moser conjecture (now proven), which concern the maximizing number of times a set in the real numbers sums to the same number.

5. Bibliography

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