The Borwein-Bailey-Plouffe formula

Tristan Liu

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1 Introduction

The BBP formula was the first formula which allowed for the computation digits of π in binary without having to compute all the digits before it. This was inspired by other similar types of formulas for numbers like $\log(2)$.

The formula has been used to verify computations of many digits of π.

In this paper, we will prove the BBP formula and other BBP-type formulas.

2 The sums

2.1 The Borwein-Bailey-Plouffe formula

Theorem 2.1. ***The Borwein-Bailey-Plouffe formula***

$$\pi = \sum_{k=0}^{\infty} \left[\frac{1}{16^k} \left(\frac{4}{8k+1} - \frac{2}{8k+4} - \frac{1}{8k+5} - \frac{1}{8k+6} \right) \right]$$

Proof. We can manipulate this infinite sum into an integral by noting that for $k < 8$

$$\sum_{i=0}^{\infty} \frac{1}{16^i(8i+k)} = 2^{k/2} \sum_{i=0}^{\infty} \frac{x^{8i+k}}{8i+k}|_0^{1/\sqrt{2}} = 2^{k/2} \int_0^{1/\sqrt{2}} x^{k-1} \sum_{i=0}^{\infty} x^{8i} dx = 2^{k/2} \int_0^{1/\sqrt{2}} \frac{x^{k-1}}{1-x^8} dx$$

Thus, our sum is equal to

$$\int_0^{1/\sqrt{2}} \frac{4\sqrt{2} - 8x^3 - 4\sqrt{2}x^4 - 8x^5}{1-x^8} dx.$$

We can clean up this integral by substituting $y = \sqrt{2}x$ to get

$$\frac{1}{\sqrt{2}} \int_0^1 \frac{4\sqrt{2} - 2\sqrt{2}y^3 - \sqrt{2}y^4 - \sqrt{2}y^5}{1 - y^8/16} dy = \int_0^1 \frac{64 - 32y^3 - 16y^4 - 16y^5}{16 - y^8} dy.$$

The numerator and denominator have a common factor of $y^2 + 2$ and $y^2 + 2y + 2$, so we can simplify to

$$\int_0^1 \frac{16 - 16y}{y^4 - 2y^3 + 4y - 4} dy = \int_0^1 \frac{4y}{y^2 - 2} dy - \int_0^1 \frac{4y - 8}{y^2 - 2y + 2} dy.$$

Evaluating the left integral easily gives, with the substitution $u = y^2 - 2$,

$$\int_{-2}^{-1} \frac{2}{u} = -\log(4)$$

The right integral can be broken up into

$$\int_0^1 \frac{4y - 4}{y^2 - 2y + 2} dy = \int_2^1 \frac{2}{u} dx = -\log(4)$$

and

$$\int_0^1 \frac{-4}{(y-1)^2+1} = 4\arctan(1-y)|_0^1 = -\pi$$

. Thus

$$\int_0^1 \frac{4y}{y^2-2}dy - \int_0^1 \frac{4y-8}{y^2-2y+2}dy = -\log(4) - (-\log(4) - \pi) = \pi$$

, which is a fairly straightforward proof. □

Other summations such as

$$\pi^2 = \sum_{i=0}^{\infty} \frac{1}{16^i}[\frac{16}{(8i+1)^2} - \frac{16}{(8i+2)^2} - \frac{8}{(8i+3)^2} - \frac{16}{(8i+4)^2} - \frac{4}{(8i+5)^2} - \frac{4}{(8i+6)^2} + \frac{2}{(8i+7)^2}]$$

can be proved similarly. They can also be used to compute the n^{th} digit of the constants in various bases. To avoid unwieldy sums, we define a more compact notation for such BBP-type formulas.

Definition 2.2.

$$P(s,b,n,A) = \sum_{k=0}^{\infty} \frac{1}{b^k} \sum_{j=1}^{n} \frac{a_j}{(kn+j)^s}$$

where s, b, n are integers and $A = (a_1, \cdots a_n)$ is a vector of integers.

2.2 BBP-type formulas for logarithms

Many logarithms have binary BBP formulas.

Proposition 2.3.

$$\log(2) = \sum_{k=1}^{\infty} \frac{1}{k2^k} = \frac{1}{2}P(1,2,1,(1))$$

and

$$\log(3) = P(1,4,2,(1,0)).$$

Proof. We know that

$$x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots = -\log(1-x)$$

which can be verified by integrating the power series for $\frac{1}{1-x}$. Evaluating at $\frac{1}{2}$ gives

$$\log(2) = \sum_{k=1}^{\infty} \frac{1}{k2^k} = \frac{1}{2}P(1,2,1,(1))$$

Evaluating at $\frac{1}{4}$ gives

$$\log(3/4) = \sum_{k=1}^{\infty} \frac{1}{k4^k}$$

We can combine these to get

$$\log(3) = 2\log(2) - \log(3/4) = 2\sum_{k=1}^{\infty} \frac{1}{k2^k} - \sum_{k=1}^{\infty} \frac{1}{k4^k} = \frac{1}{2}\sum_{k=}^{\infty} \frac{1}{4^k}(\frac{2}{2k+1} + \frac{1}{2k+2}) - \frac{1}{4}\sum_{k=0}^{\infty} \frac{1}{4^k}\frac{2}{2k+2}$$

$$= \sum_{k=0}^{\infty} \frac{1}{4^k}(\frac{1}{2k+1}) = P(1,4,2,(1,0)).$$

□

Proposition 2.4.

$$\log(5) = \frac{4}{27}P(1,3^4,4,(3^2,3,1,0)).$$

Proof. We want to evaluate

$$\frac{4}{27}\sum_{k=0}^{\infty}\frac{1}{81^k}\left(\frac{3^2}{4k+1}+\frac{3}{4k+2}+\frac{1}{4k+3}\right)$$

Consider

$$\int_0^{1/3}\frac{x^{k-1}}{1-x^4}dx=\int_0^{1/3}x^{k-1}\sum_{i=0}^{\infty}x^{4i}dx=\sum_{i=0}^{\infty}\frac{x^{4i+k}}{4i+k}|_0^{1/3}=\sum_{i=0}^{\infty}\frac{1}{3^k}\frac{1}{81^i(4i+k)}$$

Thus

$$\frac{4}{27}\sum_{k=0}^{\infty}\frac{1}{81^k}\left(\frac{3^2}{4k+1}+\frac{3}{4k+2}+\frac{1}{4k+3}\right)=4\int_0^{\frac{1}{3}}\frac{1+x+x^2}{1-x^4}dx=\int_0^{\frac{1}{3}}\frac{2x}{1+x^2}dx-\int_0^{\frac{1}{3}}\frac{3}{x-1}dx+\int_0^{\frac{1}{3}}\frac{1}{x+1}dx$$

Substituting $u=1+x^2$ for the first integral gives

$$\int_0^{\frac{1}{3}}\frac{2x}{1+x^2}dx=\int_1^{\frac{10}{9}}\frac{1}{u}du=[\log|u|]_1^{\frac{10}{9}}=\log|10/9|-\log|1|=\log(2)+\log(5)-2\log(3)$$

Evaluating the second integrals gives

$$\int_0^{\frac{1}{3}}\frac{3}{x-1}dx=[3\log|x-1|]_0^{\frac{1}{3}}=3\log|-2/3|-3\log|1|=3\log(2)-3\log(3)$$

Evaluating the last integral gives

$$\int_0^{\frac{1}{3}}\frac{1}{x+1}dx=[\log|x+1|]_0^{\frac{1}{3}}=\log|4/3|-\log|1|=2\log(2)-\log(3)$$

Combining it all gives

$$\int_0^{\frac{1}{3}}\frac{2x}{1+x^2}dx-\int_0^{\frac{1}{3}}\frac{3}{x-1}dx+\int_0^{\frac{1}{3}}\frac{1}{x+1}dx=\log(2)+\log(5)-2\log(3)-(3\log(2)-3\log(3))+2\log(2)-\log(3)=\log(5)$$

as desired. □

Binary BBP representations have been found for the logarithms for the following primes:

$$2,3,5,7,11,13,17,19,29,31,37,41,43,61,73,109,113,127,151,241,257,331,$$

$$337,397,683,1321,1429,1613,2113,2731,5419,8191,14449,26317,38737,43691,61681$$

$$65537,87211,131071,174763,246241,262657,268501,279073,312709$$

It is unknown whether there are infinitely many primes which have a BBP formula or even whether log(23) has a BBP formula.

Proposition 2.5.

$$\sqrt{5}\log(\phi)=\sum_{k=0}^{\infty}\frac{1}{5^k}\frac{1}{2k+1}=P(1,5,2,(1,0)).$$

Proof. Note showing this is equivalent to showing

$$\sum_{k=1}^{\infty}\frac{1}{(2k-1)(\sqrt{5})^{2k-1}}=\log(\phi)$$

Consider

$$\frac{1}{1-x^2}=\sum_{k=1}^{\infty}x^{2k-2}.$$

Integrating gives

$$\frac{1}{2}(\log(1+x)-\log(1-x))=\sum_{k=1}^{\infty}\frac{x^{2k-1}}{2k-1}$$

where we integrate the sum term by term and the rational function by partial fractions. Evaluating at $\frac{1}{\sqrt{5}}$ gives

$$\sum_{k=1}^{\infty}\frac{1}{(2k-1)(\sqrt{5})^{2k-1}}=\frac{1}{2}\log(\frac{1+1/\sqrt{5}}{1-1/\sqrt{5}})=\frac{1}{2}\log(\frac{(1+\sqrt{(5)})^2}{4})=\log(\phi)$$

as desired □

2.3 Other BBP-type formulas

Several formulas have also been found for arctangent including

$$\arctan(2)=\frac{1}{2^3}P(1,2^4,8,(2^3,0,2^2,0,-2,0,-1,0))$$

as well as

$\arctan(4/5)=\frac{1}{2^{17}}P(1,2^{20},40,(0,2^{19},0,-2^{17}\cdot 3,-2^{15}\cdot 3\cdot 5,0,0,2^{15}\cdot 5,0,2^{15},0,-2^{13}\cdot 3,0,0,2^{10}\cdot 5,$ $2^{11}\cdot 5,0,2^{11},0,2^{10},0,0,0,2^7\cdot 5,2^5\cdot 3\cdot 5,2^7,0,-2^5\cdot 3,0,0,0,2^3\cdot 5,0,2^3,-5,-2\cdot 3,0,0,0,0))$

Other binary BBP formulas have been found or conjectured for numbers such as

$$\log^2(2),\log^3(2),\log^4(2),\log^5(2),\zeta(3),\zeta(5),\pi^2,\pi^3,\pi^4,G.$$

Ternary BBP formulas have also been found/conjectured for numbers such as

$$\pi\sqrt{3},\pi^2,\log(2),\log(3),\log^2(3).$$

3 Finding Formulas and Digits

These formulas are largely experimentally generated by a computer search and can be used to compute digits in various bases.

When doing a computer search for BBP-type formulas, parameters values are picked for s, b, and n in $P(s,b,n,A)$. The individual sums are evaluated and PSLQ is used to find integer relations in the hopes of discovering a formula.

We consider

$$\pi=\sum_{k=0}^{\infty}\left[\frac{1}{16^k}\left(\frac{4}{8k+1}-\frac{2}{8k+4}-\frac{1}{8k+5}-\frac{1}{8k+6}\right)\right]$$

and use it to find digits of π.

We define $S_1=\sum_{k=0}^{\infty}\frac{1}{16^k(8k+1)}$. We can compute the hexadecimal digits from position $d+1$ and on by considering $\mathrm{frac}(16^d S_1)=\sum_{k=0}^{d}\frac{16^{d-k} \pmod{8k+1}}{8k+1} \pmod 1+\sum_{k=d+1}^{\infty}\frac{16^{d-k}}{8k+1} \pmod 1$ The terms of the first summation can be computed fairly quickly because modular exponentiation is fairly fast. The second summation only requires a few terms in order to get the desired precision.

Using this formula is a relatively fast way to verify accuracy when computing digits π with fast converging series, by checking later digits because errors in computations would propagate to the last digits. For example, in 2010, Alexander Yee and Shigeru Kondo computed 5 trillion digits of pi. They used two BBP-type formulas for verification:

$$\pi=\sum_{k=0}^{\infty}\left[\frac{1}{16^k}\left(\frac{4}{8k+1}-\frac{2}{8k+4}-\frac{1}{8k+5}-\frac{1}{8k+6}\right)\right]$$

as well as

$$\pi=\frac{1}{2^6}\sum_{k=0}^{\infty}\left[\frac{(-1)^k}{1024^k}\left(\frac{256}{10k+1}+\frac{1}{10k+9}-\frac{64}{10k+3}-\frac{32}{4k+1}-\frac{4}{10k+5}-\frac{4}{10k+7}-\frac{1}{4k+3}\right)\right]$$

which took around 65 hours running simultaneously on two different computers to compute 32 hexadecimal digits ending with the $4,152,410,118,610^{\text{th}}$ digit, whereas the main computation, using the Chudnovsky Formula:

$$\frac{1}{\pi} = \frac{\sqrt{10005}}{4270934400}\sum_{k=0}^{\infty}(-1)^k \frac{(6k)!}{(k!)^3(3k)!}\frac{(13591409 + 545140134k)}{640320^{3k}}$$

took about 90 days.

The quadrillionth binary digit of π was computed with this method.

This method has implications as to how quick digits of π can be calculated. Bailey, Borwein, and Plouffe then also provided methods for computing the d^{th} hexadecimal digits of π, $\log(2)$, π^2 and $\log^2(2)$ in $O(d \log^3 d)$ and $O(\log(d)$ space. This has later been shown for $G, \pi^3, \log^3(2), \zeta(3), \pi^4, \log^4(2), \log^5(2), \zeta(5)$ as well. This is an improvement on previous algorithms which typically needed had linear space complexity. It is surprising that the digits of π can be found without having to compute the digits before it and has possible implications regarding the normality of constants such as π.

4 Sources

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