Q-Series

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Introduction

<u>Definition.</u> *q-Pochhammer symbol*, *q-series*

Let $a, q \in \mathbb{C}$ and $n \in \mathbb{N}$. Define *q-Pochhammer symbol* $(a;q)_n = \prod_{k=0}^{n-1}(1-aq^k)$ for the n finite case, and let $(a;q)_\infty = \prod_{k=0}^{\infty}(1-aq^k)$ for the n infinite case. A *q-series* is a series using q-Pochhammer symbols in some way.

Examples

When we consider q-Pochhammer symbols, we usually let q be some unspecified constant, and we define our own values of a. The index n is also usually constant, but most of the time it ends up going to infinity. To get a sense for what q-Pochhammer symbols look like and how they behave, we will begin by simply plugging in some values for a and evaluating the q-Pochhammer symbol $(a;q)_n$.

Recall our definition of q-Pochhammer symbols,

$$(a;q)_n = \prod_{k=0}^{n-1}(1-aq^k) = (1-a)(1-aq)(1-aq^2)\dots(1-aq^{n-1})$$

For starters, let's plug in some very simple values for a. If we try $a = 0$, then we find

$$(0;q)_n = \prod_{k=0}^{n-1}(1-0q^k) = \prod_{k=0}^{n-1} 1 = 1$$

It also makes sense to try $a = 1$, which produces

$$(1;q)_n = \prod_{k=0}^{n-1}(1-1q^k) = \prod_{k=0}^{n-1}(1-q^k)$$
$$= (1-1)(1-q)(1-q^2)\dots(1-q^{n-1}) = 0$$

Although $a = 0$ and $a = 1$ are more trivial examples, a more interesting value to try is $a = q$.

$$(q;q)_n = \prod_{k=0}^{n-1}(1-qq^k) = \prod_{k=0}^{n-1}(1-q^{k+1}) = \prod_{k=1}^{n}(1-q^k)$$
$$= (1-q)(1-q^2)(1-q^3)\cdots(1-q^n)$$

This particular type of q-Pochhammer symbol is so special that it gets its own name, the Euler function, as well as its own notation. In this particular case, it is denoted

$$(q)_n = (q;q)_n = \prod_{k=1}^{n}(1-q^k)$$

and as $n \to \infty$, it can also be written

$$\phi(q) = (q;q)_\infty = \prod_{k=1}^{\infty}(1-q^k)$$

Theorems

The ultimate goal of this paper is to look at ways of converting between finite and infinite q-Pochhammer symbols (that is, n finite vs n infinite in $(a;q)_n$). We are interested in writing finite q-Pochhammer symbols in terms of infinite q-Pochhamer symbols, and we are also interested in writing infinite q-Pochhamer symbols in terms of finite q-Pochhammer symbols.

We begin by looking at ways to write finite q-Pochhammer symbols in terms of infinite q-Pochhammer symbols, as that will prove to be the easier direction.

<u>Theorem.</u>

$$(a;q)_n = \frac{(a;q)_\infty}{(aq^n;q)_\infty}$$

<u>Proof.</u>

We will begin with the right hand side and show that it simplifies to the left hand side. Expanding out the q-Pochhammer symbols and simplifying, we obtain

$$\frac{(a;q)_\infty}{(aq^n;q)_\infty} = \frac{\prod_{k=0}^{\infty}(1-aq^k)}{\prod_{k=0}^{\infty}(1-aq^nq^k)} = \frac{\prod_{k=0}^{\infty}(1-aq^k)}{\prod_{k=0}^{\infty}(1-aq^{n+k})}$$

Since the exponent of the bottom product is $n+k$, and the index starts at $k=0$, we can shift the index so that it starts at $k=n$, and the exponent becomes k. The upper index does not need to be changed because it is an infinite product.

$$\frac{\prod_{k=0}^{\infty}(1-aq^k)}{\prod_{k=0}^{\infty}(1-aq^{n+k})} = \frac{\prod_{k=0}^{\infty}(1-aq^k)}{\prod_{k=n}^{\infty}(1-aq^k)}$$

We then split up the top product at index n:

$$\frac{\prod_{k=0}^{\infty}(1-aq^k)}{\prod_{k=n}^{\infty}(1-aq^k)} = \frac{\prod_{k=0}^{n-1}(1-aq^k)\cdot\prod_{k=n}^{\infty}(1-aq^k)}{\prod_{k=n}^{\infty}(1-aq^k)}$$

The right product in the numerator then cancels with the denominator, and we're left with

$$\frac{\prod_{k=0}^{n-1}(1-aq^k)\cdot\prod_{k=n}^{\infty}(1-aq^k)}{\prod_{k=n}^{\infty}(1-aq^k)} = \prod_{k=0}^{n-1}(1-aq^k)$$

This is the definition of the q-Pochhammer symbol $(a;q)_n$. Putting this all together, we recover

$$\frac{(a;q)_\infty}{(aq^n;q)_\infty} = \prod_{k=0}^{n-1}(1-aq^k) = (a;q)_n$$

This proves the theorem. □

We now discuss with a very important theorem in q-series theory, proven by Cauchy and in parts by Euler. It is known as the q-analogue of the binomial theorem, or the q-binomial theorem for short. This theorem provides the framework for us to write infinite q-Pochhammer symbols as a power series using finite q-Pochhammer symbols as coefficients—in other words, a q-series.

Theorem. *Q-Binomial Theorem*

For $|q|, |z| < 1$,

$$\sum_{n=0}^{\infty}\frac{(a;q)_n}{(q;q)_n}z^n = \frac{(az;q)_\infty}{(z;q)_\infty}$$

Proof.

In this proof, we will consider a series expansion for the right hand side of the equation, and we will show that the coefficients of that series expansion correspond to that of the power series on the left hand side of the equation. But in order to be able to think about a series expansion for the right hand side, we first need to show that the products

$$(az;q)_n = \prod_{k=0}^{n-1}(1-azq^k)$$

$$(z;q)_n = \prod_{k=0}^{n-1}(1 - zq^k)$$

converge as $n \to \infty$. In the lower product,

$$|z| < 1$$

$$|q| < 1$$

$$|zq^k| < 1$$

$$|1 - zq^k| < 1$$

We know that if all the terms in a product have magnitude less than 1, then the product must converge (not necessarily to 0). Therefore,

$$\lim_{n\to\infty}(z;q)_n = \prod_{k=0}^{n-1}(1 - zq^k) = A$$

To show the first product converges, it is essentially the same as the second product, only with an extra a constant. No matter how big of an a we pick, since $|q| < 1$, we can always raise q to some power k_1 so that

$$|q^{k_1}| < \frac{1}{|a|}$$

$$|a||q^{k_1}| < 1$$

$$|aq^{k_1}| < 1$$

Since $|z| < 1$, multiplying this product by z should still preserve its magnitude as less than 1,

$$|azq^{k_1}| < 1$$

The magnitude will still be less than 1 no matter how many more q's we multiply it by, since $|q| < 1$. Hence, for any exponent $k \geq k_1$,

$$|azq^k| < 1$$

Finally, we return to our product, and we split it up at k_1:

$$(az;q)_n = \prod_{k=0}^{n-1}(1 - aq^k) = \prod_{k=0}^{k_1-1}(1 - aq^k) \cdot \prod_{k=k_1}^{n-1}(1 - aq^k)$$

Since all of the terms in the right piece of the product have magnitude less than 1, then the right product must converge as $n \to \infty$:

$$\lim_{n\to\infty}\prod_{k=k_1}^{n-1}(1 - aq^k) = B$$

Additionally, the left piece of the product is a finite product, hence

$$\prod_{k=0}^{k_1-1}(1-aq^k)=C$$

So putting this together, we have

$$\lim_{n\to\infty}(az;q)_n=\lim_{n\to\infty}\prod_{k=0}^{n-1}(1-aq^k)=\lim_{n\to\infty}[\prod_{k=0}^{k_1-1}(1-aq^k)\cdot\prod_{k=k_1}^{n-1}(1-aq^k)]$$

$$=\prod_{k=0}^{k_1-1}(1-aq^k)\cdot\lim_{n\to\infty}\prod_{k=k_1}^{n-1}(1-aq^k)=BC$$

Thus the first product also converges.

Since the limits of these products converge, they are nonzero. Hence their quotient is also well-defined and converges:

$$\frac{(az;q)_\infty}{(z;q)_\infty}=\lim_{n\to\infty}\frac{(az;q)_n}{(z;q)_n}=\frac{\lim_{n\to\infty}(az;q)_n}{\lim_{n\to\infty}(z;q)_n}=\frac{BC}{A}$$

Since our quotient converges (A, B, and C are most likely functions of some sort), then it is analytic on the disc $|z|<1$. Because it is analytic, we can write it as some power series which will converge to the correct value:

$$\frac{(az;q)_\infty}{(z;q)_\infty}=\sum_{n=0}^{\infty}A_nz^n$$

For ease of referencing later on, we will use function F to denote this quotient and its corresponding sum.

$$F(z)=\frac{(az;q)_\infty}{(z;q)_\infty}=\sum_{n=0}^{\infty}A_nz^n$$

Now we would like to know what the A_n coefficients of function F are. To do this, we first write a recursive definition for F:

$$(1-z)F(z)=(1-az)F(qz)$$

To verify this recursive identity, we just expand using the definition of F, and we cut off the first terms of the products:

$$(1-z)F(z)=(1-z)\frac{(az;q)_\infty}{(z;q)_\infty}$$

$$=(1-z)\frac{\prod_{k=0}^{\infty}(1-azq^k)}{\prod_{k=0}^{\infty}(1-zq^k)}$$

$$= (1-z)\frac{(1-az)\cdot\prod_{k=1}^{\infty}(1-azq^k)}{(1-z)\cdot\prod_{k=1}^{\infty}(1-zq^k)}$$

$$= (1-az)\frac{\prod_{k=1}^{\infty}(1-azq^k)}{\prod_{k=1}^{\infty}(1-zq^k)}$$

$$= (1-az)\frac{\prod_{k=0}^{\infty}(1-azq^{k+1})}{\prod_{k=0}^{\infty}(1-zq^{k+1})}$$

$$= (1-az)\frac{\prod_{k=0}^{\infty}(1-(aqz)q^k)}{\prod_{k=0}^{\infty}(1-(qz)q^k)}$$

$$= (1-az)\frac{(aqz;q)_\infty}{(qz;q)_\infty} = (1-az)F(qz)$$

Substituting in our power series expansion for $F(z)$, we obtain

$$(1-z)\sum_{n=0}^{\infty} A_n z^n = (1-az)\sum_{n=0}^{\infty} A_n (qz)^n$$

Expanding,

$$(1-z)(A_0 + A_1 z + A_2 z^2 + \ldots) = (1-az)(A_0 + A_1 qz + A_2 q^2 z^2 + \ldots)$$

$$A_0 + (A_1 - A_0)z + (A_2 - A_1)z^2 + \ldots = A_0 + (qA_1 - aA_0)z + (q^2 A_2 - aqA_1)z^2 + \ldots$$

Equating coefficients with the same z^n term, we have

$$A_0 = A_0$$
$$A_1 - A_0 = qA_1 - aA_0$$
$$A_2 - A_1 = q^2 A_2 - aqA_1$$
$$\ldots$$
$$A_n - A_{n-1} = q^n A_n - aq^{n-1} A_{n-1}$$

Solving for A_n, we obtain a recursive definition for the A coefficients:

$$A_n - q^n A_n = A_{n-1} - aq^{n-1} A_{n-1}$$
$$(1-q^n)A_n = (1-aq^{n-1})A_{n-1}$$
$$A_n = \frac{1-aq^{n-1}}{1-q^n} A_{n-1}$$

Now, we want to evaluate A_0. Since $F(z) = \sum_{n=0}^{\infty} A_n z^n$, then if we plug in $z = 0$,

$$F(0) = \sum_{n=0}^{\infty} A_n (0)^n = A_0$$

We can evaluate $F(0)$ using our other definition of $F(z)$:

$$F(0) = \frac{(0z;q)_\infty}{(0;q)_\infty} = \frac{(0;q)_\infty}{(0;q)_\infty}$$

When we plugged in values from earlier, we observed that $(0;q)_n = 1$. Hence, $(0;q)_\infty = \lim_{n\to\infty}(0;q)_n = 1$, so

$$F(0) = \frac{(0;q)_\infty}{(0;q)_\infty} = \frac{1}{1} = 1$$

Hence

$$A_0 = 1$$

We now have a recursive formula for A_n in terms of A_{n-1}, as well as a definition for A_0. Thus, we can evaluate the coefficients:

$$A_0 = 1$$
$$A_1 = \frac{1-aq^0}{1-q^1}A_0 = \frac{(1-a)}{(1-q)}$$
$$A_2 = \frac{1-aq^1}{1-q^2}A_1 = \frac{1-aq}{1-q^2}\cdot\frac{1-a}{1-q} = \frac{(1-a)(1-aq)}{(1-q)(1-q^2)}$$
$$A_3 = \frac{1-aq^2}{1-q^3}A_2 = \frac{1-aq^2}{1-q^3}\cdot\frac{(1-a)(1-aq)}{(1-q)(1-q^2)} = \frac{(1-a)(1-aq)(1-aq^2)}{(1-q)(1-q^2)(1-q^3)}$$
$$\cdots$$
$$\begin{aligned} A_n &= \frac{(1-a)(1-aq)(1-aq^2)\cdots(1-aq^{n-1})}{(1-q)(1-q^2)(1-q^3)\cdots(1-q^n)} = \frac{\prod_{k=0}^{n-1}(1-aq^k)}{\prod_{k=1}^{n}(1-q^k)} \\ &= \frac{\prod_{k=0}^{n-1}(1-aq^k)}{\prod_{k=0}^{n-1}(1-qq^{k-1})} \\ &= \frac{(a;q)_n}{(q;q)_n} \end{aligned}$$

Therefore, our series looks like

$$\frac{(az;q)_\infty}{(z;q)_\infty} = F(z) = \sum_{n=0}^{\infty} A_n z^n = \sum_{n=0}^{\infty}\frac{(a;q)_n}{(q;q)_n}z^n$$

This proves the theorem. □

The q-Binomial Theorem is very powerful, and is responsible for many identities involving q-Pochhammer symbols. To demonstrate its power, we will demonstrate a few identities which follow from the q-Binomial Theorem or methods similar to the q-Binomial Theorem. These identities demonstrate how to write an infinite q-Pochhammer symbol in its series expansion of finite q-Pochhammer symbols.

Theorem (Corollary).

For $|z| < 1$,

$$\frac{1}{(z;q)_\infty} = \sum_{n=0}^{\infty} \frac{1}{(q;q)_n} z^n$$

Proof.

The proof is given by taking the statement of the q-binomial theorem and letting $a = 0$, using the fact that $(0;q)_n = 1$:

$$\frac{(az;q)_\infty}{(z;q)_\infty} = \sum_{n=0}^{\infty} \frac{(a;q)_n}{(q;q)_n} z^n$$

$$a = 0$$

$$\frac{(0z;q)_\infty}{(z;q)_\infty} = \sum_{n=0}^{\infty} \frac{(0;q)_n}{(q;q)_n} z^n$$

$$\frac{(0;q)_\infty}{(z;q)_\infty} = \sum_{n=0}^{\infty} \frac{(0;q)_n}{(q;q)_n} z^n$$

$$\frac{1}{(z;q)_\infty} = \sum_{n=0}^{\infty} \frac{1}{(q;q)_n} z^n$$

This proves the theorem. □

Theorem (Corollary).

For $|z| < \infty$,

$$(z;q)_\infty = \sum_{n=0}^{\infty} \frac{(-1)^n q^{\frac{n(n-1)}{2}}}{(q;q)_n} z^n$$

Proof.

This is a corollary of the q-binomial theorem in the sense that we will use a very similar method to prove it: *q-difference equations.* The idea behind the q-difference equations method, which is frequently used to prove q-series identities in general, is that we find some kind of functional equation relating

$F(z)$ and $F(qz)$. Then, using some base case $F(0)$ or something similar, we iteratively apply and substitute the functional equations to produce some kind of closed formula. Since these functional equations will involve products most of the time, when iterated the closed formula will usually include some kind of product, which then gets turned into a q-Pochhammer symbol.

In this proof, we will try to express the left hand side as some kind of power series, and then we will evaluate the coefficients of this power series. Define function F to denote the q-Pochhammer symbol on the left side:

$$F(z) = (z;q)_\infty = \prod_{k=0}^{\infty}(1 - zq^k) = (1-z)(1-zq)(1-zq^2)\dots$$

Now, we create our recurrence. $F(z)$ and $F(qz)$ are related by the following q-difference equation:

$$F(z) = (1-z)F(qz)$$

We won't go through all of the rigorous analysis again, so this time we will just assume that $F(z)$ may be represented by some power series

$$F(z) = \sum_{n=0}^{\infty} A_n z^n$$

Substituting this power series into our recursive equation, we find

$$\sum_{n=0}^{\infty} A_n z^n = (1-z)\sum_{n=0}^{\infty} A_n (qz)^n = (1-z)\sum_{n=0}^{\infty} A_n q^n z^n$$

Expanding and writing out a few terms, we see

$$\begin{aligned} A_0 + A_1 z + A_2 z^2 + \dots &= (1-z)(A_0 + A_1 qz + A_2 q^2 z^2 + \dots) \\ &= A_0 + (A_1 q - A_0)z + (A_2 q^2 - A_1 q)z^2 + \dots \end{aligned}$$

Equating coefficients with the same degree,

$$\begin{gathered} A_0 = A_0 \\ A_1 = A_1 q - A_0 \\ A_2 = A_2 q^2 - A_1 q \\ \dots \\ A_n = A_n q^n - A_{n-1} q^{n-1} \end{gathered}$$

Solving for A_n in terms of A_{n-1},

$$A_n - A_n q^n = -A_{n-1} q^{n-1}$$

$$(1 - q^n)A_n = -A_{n-1}q^{n-1}$$
$$A_n = -\frac{q^{n-1}}{1 - q^n}A_{n-1}$$

Our first term A_0 is given by

$$A_0 = F(0) = (0; q)_\infty = 1$$

Hence, solving for terms recursively,

$$A_0 = 1$$
$$A_1 = -\frac{1}{1-q}A_0 = -\frac{1}{1-q} \cdot 1 = -\frac{(1)}{(1-q)}$$
$$A_2 = -\frac{q}{1-q^2}A_1 = -\frac{q}{1-q^2} \cdot -\frac{(1)}{(1-q)} = \frac{(1)(q)}{(1-q)(1-q^2)}$$
$$A_3 = -\frac{q^2}{1-q^3}A_2 = -\frac{q^2}{1-q^3} \cdot \frac{(1)(q)}{(1-q)(1-q^2)} = -\frac{(1)(q)(q^2)}{(1-q)(1-q^2)(1-q^3)}$$
$$\cdots$$
$$A_n = (-1)^n \frac{(1)(q)(q^2)\cdots(q^{n-1})}{(1-q)(1-q^2)(1-q^3)\cdots(1-q^n)} = (-1)^n \frac{q^{0+1+2+\cdots n-1}}{\prod_{k=1}^{n}(1-q^k)}$$

The exponent on the top of the expression is a triangular number T_{n-1}. Using the formula $T_n = \frac{n(n+1)}{2}$, or in this case $T_{n-1} = \frac{n(n-1)}{2}$, we may simplify further:

$$A_n = (-1)^n \frac{q^{0+1+2+\cdots n-1}}{\prod_{k=1}^{n}(1-q^k)} = (-1)^n \frac{q^{T_{n-1}}}{\prod_{k=0}^{n-1}(1-q \cdot q^k)} = (-1)^n \frac{q^{\frac{n(n-1)}{2}}}{(q;q)_n}$$

Finally, plugging these values of A_n back into our series expansion of $F(z)$, we recover the desired identity:

$$(z;q)_\infty = F(z) = \sum_{n=0}^{\infty} \frac{(-1)^n q^{\frac{n(n-1)}{2}}}{(q;q)_n} z^n$$

This completes the proof. $\square$

References

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