EULER'S PENTAGONAL NUMBER THEOREM

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Abstract. In this paper, we will discuss the history of Euler's Pentagonal Number Theorem and the different ways numerous mathematicians have proved it, such as Franklin's proof. Additionally, we will elaborate on Jacobi's triple product formula.

1. Pentagonal Numbers

Pentagonal numbers get their name because they are numbers that can be shown as a pentagonal number of dots. However, there is a more precise definition:

Definition 1.1. The *pentagonal numbers* are all numbers of the form $\frac{3n^2-n}{2}$, where $n \geq 1$.

However, for the Pentagonal Number Theorem, we'll need something a bit more general:

Definition 1.2. The *generalized pentagonal numbers* are all numbers of the form $\frac{3n^2-n}{2}$ where n is taking values from the sequence $0, 1, -1, 2, -2, 3, -3, 4, \ldots$

In this paper, we will denote the pentagonal numbers as w_k. This is a simple definition and it works quite well for us.

2. Partitions

Partitions are used widely in Franklin's proof of Euler's pentagonal number theorem.

Definition 2.1. A *partition* of a number$n$ is a representation of n as a sum of positive integers. Order does not matter.

For example 4 has 5 partitions: 4, 3 + 1, 2 + 2, 2 + 1 + 1, 1 + 1 + 1 + 1. Euler's pentagonal number theorem implies a recurrence for finding $p(n)$, where $p(n)$ is the amount of partitions of n:

$$p(n) = \sum_k = (-1)^{k-1} \cdot p(n - g_k).$$

where k takes the value of all nonzero integers and g_k is the kth generalized pentagonal number.

There is an identity by Euler we will prove:

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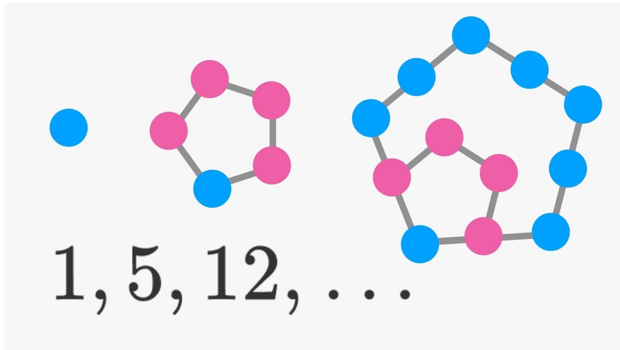

Figure 1. The pentagonal numbers

Theorem 2.2. *Let x be a real number, then*

$$(1 + x + x^2 + \dots)(1 + x^2 + x^4 + \dots)(1 + x^3 + x^6 + \dots)(\dots) = \sum p(n)x^n$$

Proof. For this identity we will use the distributive law: to multiply $(a+b)(c+d)$, we choose one of a and b from column A. Then, we choose on of c and d from column B. After that, you add up the possibilities. In this example we can get $ac+ad+bc+bd$. We can apply this trick to larger products too. So too evaluate this, we can apply the same trick in the following manner: we choose x^{k_1} from column A, x^{2k_2} from column B, x^{3k_3} from column C, on and on. After that we multiply them, to get $x^{k_1+2k_2+3k_3+sk_s}$, and then add up the possibilities to get

$$\sum_{k_1,k_2,\dots} x^{k_1+2k_2+3k_3+sk_s}$$

This is a sum of the powers of x, so we can get

$$\sum_{k_1,k_2,\dots} x^n.$$

However, the term x^n will occur as many times as we can write $n = k_1 + 2k_2 + \dots + sk_s$. This is a fancy way to write a partition, so the sum becomes $\sum p(n)x^n$. ■

Here is another theorem to compute partitions:

Theorem 2.3.

$$\frac{1}{(1-x)(1-x^2)(1-x^3)\dots} = \sum p(n)x^n$$

Proof. This follows quite easily from the first formula: we take inverses of the series on the left side, and we are done. ■

3. Euler's Pentagonal Number Theorem

Now, lets move onto one of Euler's most profound discoveries, the Pentagonal Number Theorem. For more on the history refer to [Bel10].

Theorem 3.1. *The Pentagonal Number Theorem states that*

$$\prod_{n=1}^{\infty}(1-x^n) = \sum_{k=-\infty}^{\infty}(-1)^k x^{k(3k-1)/2} = 1+\sum_{k=1}^{\infty}(-1)^k \cdot (x^{k(3k+1)/2}+x^{k(3k-1)/2}).$$

Notice the $\frac{k(3k+1)}{2}$ and the $\frac{k(3k-1)}{2}$ in the exponents of the x's in the sum. So in simpler terms, this means that $(1-x)(1-x^2)(1-x^3)\cdots = 1-x-x^2+x^5+x^7-x^{12}-x^{15}+x^{22}+x^{26}$, where the exponents for the right hand side are the generalized pentagonal numbers.

There are many proofs of this theorem but we'll look at two: Euler's original proof, and Franklin's extraordinarily simple proof.

Proof. We first let $A_0 = \prod_{k=1}^{\infty}(1-z^k)$. We will use the identity

$$\prod_{k=1}^{N}(1-a_k) = 1-a_1-\sum_{k=2}^{N} a_k(1-a_1)\dots(1-a_{k-1}).$$

We can prove this by induction. Then, we can use the identity with $a_k = z^k$ and $N=\infty$ to get

$$A_0 = 1-z-\sum_{k=2}^{\infty} z^k(1-z)\dots(1-z^{k-1})$$

$$= 1-z-\sum_{k=0}^{\infty} z^{k+2}(1-z)\dots(1-z^{k+1}).$$

Then, for $n \geq 1$ let $A_n = \sum_{k=0}^{\infty} z^{nk}(1-z^n)\dots(1-z^{n+k})$. So, $A_0 = 1-z-z^2A_1$, and for $n \geq 1$ we have

$$A_n = 1-z^n+\sum_{k=1}^{\infty} z^{nk}(1-z^n)\dots(1-z^{n+k})$$

$$= 1-z^n+\sum_{k=1}^{\infty} z^{nk}(1-z^{n+1})\dots(1-z^{n+k})-\sum_{k=1}^{\infty} z^{n(k+1)}(1-z^{n+1})\dots(1-z^{n+k})$$

$$= 1-z^n+z^n(1-z^{n+1})+\sum_{k=2}^{\infty} z^{nk}(1-z^{n+1})\dots(1-z^{n+k})-\sum_{k=1}^{\infty} z^{n(k+1)}(1-z^{n+1})\dots(1-z^{n+k})$$

$$= 1-z^{2n+1}+\sum_{k=0}^{\infty} z^{n(k+2)}(1-z^{n+1})\dots(1-z^{n+k+2}-\sum_{k=0}^{\infty} z^{n(k+2)}-(1-z^{n+1})\dots(1-z^{n+k+1})$$

$$= 1-z^{2n+1}-\sum_{k=0}^{\infty} z^{n(k+2)+n+k+2}(1-z^{n+1})\dots(1-z^{n+k+1})$$

$$= 1-z^{2n+1}-z^{3n+2}\sum_{k=0}^{\infty} z^{(n+1)k}(1-z^{n+1})\dots(1-z^{n+k+1})$$

$$= 1-z^{2n+1}-z^{3n+2}A_{n+1}$$

Figure 2. The first diagram shows the partition of 20 into $7+6+4+3$. The second diagram shows that result of the operation.

Figure 3. The first diagram shows when $m = s$ and the rightmost diagonal and bottom row meet, and the second diagram shows if we attempted to move the rows back.

We can then check by induction that for all M,

$$A_0 = 1 - z + \sum_{n=1}^{M}(-1)^n(z^{n(3n+1)/2} - z^{(n+1)(3n+2)/2}) + (-1)^{M+1}z^{(M+1)(3M+2)/2}A_{m+1}.$$

Taking $M = \infty$ gives us the pentagonal number theorem. ∎

Now that we have looked at Euler's proof, let's look at Franklin's. Franklin's involves a Ferrers diagram of any partition of a number n into distinct parts.

Proof. Let m be the number of elements in the smallest row of the diagram, and then let s be the number of elements in the rightmost 45 degree line.

Take the right most 45 degree line and move it to form a new row, as in Figure 2 where $n = 20$ and we are using the partition $20 = 7+6+4+3$.

If $m \leq s$, we can reverse the process by moving the elements of the last line to the first m rows, and we get back where we started. If we do this, we always change the number of rows and when we do the process again, we get back where we started. This enables us to pair off Ferrers diagrams contributing 1 and -1 to the x^n term of the series, resulting in a net coefficient of 0. There are only 2 cases in which this does not happen:

(1) The bottom and the right most row meet and $m = s$

To see how it looks and the result of the operation, refer to figure 3. We do the operation and this does not change the amount of rows, so doing it again would not

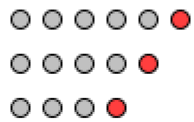

Figure 4. When $m = s + 1$ and the rightmost diagonal and bottom row meet

take us back to the original diagram. If there are m elements in the last row of the first diagram,

$$n = m + (m+1) + (m+2) + \cdots + (2m-2) = \frac{m(3m-1)}{2} = \frac{k(3k-1)}{2},$$

where $k = m$. Note that the sign associated with this partition is $(-1)^s$, which by construction equals $(-1)^m$ and $(-1)^k$.

(2) When $m = s + 1$ and the rightmost diagonal and bottom row meet

We would want to move the right diagonal to the bottom row, shown in Figure 4, but that would give us two rows with the same amount of elements, which is not allowed because we are counting partitions into distinct parts. Since this is the previous case, but we have one fewer row, so we have

$$n = m + (m+1) + (m+2) + \cdots + (2m-2) = \frac{(m-1)(3m-2)}{2} = \frac{k(3k-1)}{2}$$

where $k = 1 - m$. Here the associated sign is $(-1)^s$ with $s = m - 1 = -k$, therefore the sign is again $(-1)^k$, so we are done.

■

4. Jacobi's Triple Product Formula

It actually turns out that Euler's Pentagonal Number Theorem is just a special case of Jacobi's Triple Product Identity.

Theorem 4.1. *(Jacobi's Triple Product Identity): If $z \neq 0$ and $|x| < 1$,*

$$\prod_{n=0}^{\infty}(1 - x^{2n+2})(1 + x^{2n+1}z)(1 + x^{2n+1}z^{-1}) = \sum_{n=-\infty}^{\infty} x^{n^2} z^n.$$

There are also multiple proofs for this identity, but we'll do George E. Andrews' extremely short proof. However, this uses 2 identities, which are easily verified.

Proof. First, let's establish the two identities that we will need.

Lemma 4.2. *Given $|x| < 1$,*

$$\prod_{n=0}^{\infty}(1 + x^n z) = \sum_{n=0}^{\infty} \frac{x^{n(n-1)/2} \cdot z^n}{(1-x)\ldots(1-x^n)}.$$

And now for the second one:

Lemma 4.3. *Given* $|x| < 1$ *and* $|z| < 1$*,*

$$\prod_{n=0}^{\infty}(1 + x^n z)^{-1} = \sum_{n=0}^{\infty} \frac{(-1)^n \cdot z^n}{(1-x)\dots(1-x^n)}.$$

We can use these identities to our advantage:

$$\prod_{n=0}^{\infty}(1 + x^{2n+1} \cdot z) = \sum_{n=0}^{\infty} \frac{x^{n^2} \cdot z^n}{(1-x^2)\dots(1-x^{2n})},$$

by the 1st identity. Simplifying further, we get

$$\sum_{n=0}^{\infty} \frac{x^{n^2} \cdot z^n \cdot \prod_{j=0}^{\infty}(1 - x^{2n+2j+2})}{\prod_{j=0}^{\infty}(1-x^{2j+2})},$$

$$= \frac{1}{\prod_{j=0}^{\infty}(1-x^{2j+2})} \cdot \sum_{n=-\infty}^{\infty} x^{n^2} \cdot z^n \cdot \prod_{j=0}^{\infty}(1 - x^{2n+2j+2}),$$

where all the terms with the negative n are 0. Then, by using identity 1 again, we get

$$\frac{1}{\prod_{j=0}^{\infty}(1-x^{2j+2})} \cdot \sum_{n=-\infty}^{\infty} (x^{n^2} \cdot z^n) \cdot \sum_{m=0}^{\infty} \frac{(-1)^m \cdot x^{m^2+m+2nm}}{(1-x^2)\dots(1-x^{2m})},$$

$$= \frac{1}{\prod_{j=0}^{\infty}(1-x^{2j+2})} \cdot \sum_{m=0}^{\infty} \frac{(-1)^m \cdot (xz^{-1})^m}{(1-x^2)\dots(1-x^{2m})} \sum_{n=-\infty}^{\infty} x^{(n+m)^2} z^{n+m},$$

$$= \frac{1}{\prod_{j=0}^{\infty}(1-x^{2j+2})} \prod_{j=0}^{\infty}(1 + x^{2j+1}z^{-1})^{-1} \sum_{n=-\infty}^{\infty} x^{n^2} z^n,$$

by identity 2 and replacing $n + m$ with n. This argument is valid if $|x| < |z|$. For all nonzero z, we can use analytic continuation. ∎

References

[Bel10] Jordan Bell. A summary of euler's work on the pentagonal number theorem. *Archive for history of exact sciences*, 64(3):301–373, 2010.