Theta Functions

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Abstract

In this article, we reintroduce the idea of theta functions, noting their properties and their role in the broader scope of mathematics.

1 Introduction and Definitions

We say that an integer m is "represented" by an integer valued quadratic form if there is a solution, with quadratic form q, that satisfies $q(x_1, x_2...) = m$. Then, let

$$r_q(m) := \{\vec{x} \in \mathbb{Z}^n : q(\vec{x}) = m\}$$

or, the number of representations of m in q. Here, we define the Theta Function of q as the fourier series expansion for the r_q, or, in this case:

$$\theta_q(z) := \sum_{n=0}^{\infty} r_q(m) e^{2\pi i m z}$$

From here on out, we will seek to understand the properties and symmetries of this, essentially generating function, and with that, derive information about these representation numbers.

1.1 Some Symmetries

The theta function has quite a few interesting symmetries that are worth mentioning. As being a fourier series, it is invariant under the transformation $z \to z + 1$, but this holds for any fourier series. In particular, the theta function satisfies:

$$\theta_q(z) = \sum_{n=0}^{\infty} r_q(m) e^{2\pi i m z} = \sum_{\vec{x} \in \mathbb{Z}^n} e^{2\pi i q(\vec{x}) z}$$

We can see this using Poisson Summation, which we recall here:

Theorem 1.1. *(Poisson Summation) Suppose that $f(\vec{x})$ is a Schwartz function on $\mathbb{R}^n$, or that which decays faster than any polynomial as $x \to \infty$,then f satisfies the equality :*

$$\sum_{\vec{x} \in \mathbb{Z}^n} f(\vec{x}) = \sum_{\vec{x} \in \mathbb{Z}^n} \widehat{f}(\vec{x})$$

and the sums on both sides are convergent, where

$$\widehat{f}(\vec{x}) := \int_{\vec{t} \in \mathbb{R}^n} f(\vec{t}) e^{-2\pi i \vec{x} \vec{t}}$$

is the Fourier Transform of f.

Now, one thing to notice is that since $e^{-\pi x^2}$ is its own fourier transform, Poisson summation can allow us to, with some rescaling, map a term to itself. Thus, consider the transformation $z \to \frac{-1}{Nz}$ for some natural N. For example. consider the case where $q(x) = x^2$, then take $N = 4$, to yield:

$$\theta_{x^2}(\frac{-1}{4z}) = \sqrt{-2iz} \cdot \theta_{x^2}(z), \theta_{x^2}(z) = \theta_{x^2}(z+1)$$

by Poisson Summation.

1.2 Modular Forms

Definition 1.2. We define a modular form of weight k, level N, Dirichlet character χ and multiplier system ϵ to be a holomorphic function such that

$$f(\frac{az+b}{cz+d}) = \epsilon(\gamma, k)\chi(d)(cz+d)^k f(z)$$

for all $\gamma := [[a, b], [c, d]]$. A Dirichlet Character χ is defined according to the following axioms: There exists p such that $\chi(n+p) = \chi(n)$ for all n, $\chi(xy) = \chi(x)\chi(y)$, and if $\gcd(n, k) = 1, \chi(n) = 0$, and $\chi(n) \neq 0$ if not. A holomorphic function is a complex function such that the function is differentiable in a neighborhood around any point in its domain, and a multiplier system is simply a factor that we associate with modular forms.

We notice that taking $\gamma = [[1, 1], [0, 1]]$ gives us $f(z+1) = f(z)$, and thus with this periodicity, and modular form f can be written as a Fourier Series:

$$f(z) = \sum_{n=0}^{\infty} a(n)e^{2\pi inz}$$

where the Fourier coefficients $a(n) \in \mathbb{C}$. From these, we have another elementary corollary:

Corollary 1.3. *$\theta_q(z)$ is a modular form of weight $n/2$ and level N, with character $\chi(g) = \frac{(-1)^{[n/2]}\det(q)}{g}$ (with $[k]$ being the floor), under the system $\epsilon(\gamma, k) := 1$ when n is even, and $\epsilon(\gamma, k) := \epsilon_d^{-1}(\frac{c}{d})$ where $\epsilon_d = \begin{cases} 1 & d \equiv 1 \pmod 4 \\ i & d \equiv 3 \pmod 4 \end{cases}$ and $(\frac{c}{d})$ is the standard quadratic character.*

Proof. This is evident since $\epsilon_d^2 = (\frac{-1}{d})$, so $(\epsilon_d^{-1}(\frac{c}{d}))^n = (\frac{-1}{d})^{[n/2]}) \cdot \begin{cases} \epsilon_d^{-1}(\frac{c}{d}) & \text{if n is odd,} \\ 1 & \text{if n is even} \end{cases}$ and thus we're done. □

In general, since our definitions are in the upper half plane $\mathcal{H}$, if we look at the actions of $\Gamma_0(N)$ on $\mathcal{H}$ under linear transformations of the form $z \to \frac{az+b}{cz+d}$, for $N = 1$, $\Gamma_0(N) = \mathrm{SL}_2(\mathbb{Z})$, the special linear subgroup of integer valued matrices, and so the known domain $\mathcal{F}$ is defined as:

$$\mathcal{F} := \{z \in \mathcal{H} | |z| \geq 1 \text{ and } |\mathrm{Re}(z)| \leq \frac{1}{2}\}$$

However, our domain is not compact, but can be extended to a compact surface with the addition of a point ∞ or $i\infty$ which is essentially the limit point of the upper half-plane, which we can imagine to be rendered at the point at infinity at the top of the y-axis. This point is called a cusp of $\Gamma_0(1)$, and in general, $\Gamma_0(N)$ will have a finite index in $\mathrm{SL}_2(\mathbb{Z})$, and thus its domain is the union of translates of $\mathcal{F}$, which can be made compact with the addition of finitely many cusps of $\Gamma_0(N)$.

1.3 Eisenstein Series

Cusps are quite relevant in the theory of theta functions and modular forms. We can construct a subspace of the modular forms of being the forms with vanish at the cusps, denoted as cusp forms. We can also contruct another form $E_{\mathcal{C}}(z)$ to associated with each specific cusp $\mathcal{C}$ which has value 1 at $\mathcal{C}$ and 0 at all other cusps. We call these functions and the space formed by these functions associated to cusps as the Eisenstein Series. For the sake of brevity, it is impossible to show many results for the scope of this paper and its intended audience, but we cite some results from the range of study on these Series, as they are understood quite well. For example, they can actually be understood in an explicit fashion, as is with the cusp $\mathcal{C} = i\infty$ in $\mathrm{SL}_2(\mathbb{Z})$, the Eisenstein series of weight $k \in 2\mathbb{Z} > 2$, is given by :

$$E_k(z) := \sum_{\substack{(c,d)\in\mathbb{Z}^2\\ \gcd(c,d)=1}} \frac{1}{(cz+d)^k} = 1 - \frac{2k}{B_{2k}} \sum_{m\geq 1} \sigma_{k-1}(m) q^m$$

where B_{2k} is the $2k$th Bernoulli number, $\sigma_{k-1} = \sum_{0<d|m} d^{k-1}$ is the sum of the powers of the divisors, and $q = e^{2\pi i z}$. [1]

1.4 Facts about Modular Forms

Here, we state prior results on the topic of modular forms, such that they will help us in our asymptotic statements. The space M_k of modular forms can be decomposed uniquely into a direct sum of Eisenstein series and cusp forms.
Furthermore, any Eisenstein series has Fourier coefficients $a_E(m)$ which can be as large as $c_\epsilon m^{k-1+\epsilon}$ for any $\epsilon > 0$ and some constant $c_\epsilon \in \mathbb{R} > 0$.
Any cusp form has fourier coefficients $a_C(m)$ which are no larger than $c_\epsilon m^{k/2+\epsilon}$ for any $\epsilon > 0$ and some constant $c_\epsilon \in \mathbb{R} > 0$.

1.5 Asymptotic statements about the representation numbers

Armed with the previous section, we can write our theta function as:

$$\theta_q(z) = E(z) + C(z)$$

where $E(z)$ is an Eisenstein series and $C(z)$ is a cusp form. Comparing fourier coefficients leads to:

$$r_q(m) = a_E(m) + a_C(m)$$

By our bounds on a_E, a_C, we know that if the Eisenstein Fourier coefficients are nonzero, then r_q is nonzero as well, and so m is represented by q is m is large enough. [1]

References

[1] T. Miyake, “Modular forms.” Springer Monographs in Mathematics., 2006.

[2] J. Hanke, “Quadratic forms and automorphic forms,” 2011.

[3] O. T. O’Meara, “Introduction to quadratic forms,” 1971.