EULER CIRCLE PAPER: THE J-FUNCTION

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ABSTRACT. The j-function is an important function relating to many mathematical concepts such as elliptic curves and abstract algebra. I will be referencing "Ramanujan and the modular j-invariant"[BC99], "Planar trees, free nonassociative algebras, invariants, and elliptic integrals"[DH07], "The j-Function and the Monster"[Sch10], "Is $e^{\pi\sqrt{163}}$ odd or even?"[B.S], "A Book of Abstract Algebra"[Pin10]. Throughout the paper, I will paraphrase/cite definitions and theorems from the latter articles. We will take a close look at some interesting properties of the j-function, such as its use in Monster groups, various uses in abstract algebra, and in showing that $e^{\pi\sqrt{163}}$ is very close to an integer.

1. INTRODUCTION TO THE J-FUNCTION, MONSTER GROUPS, AND ABSTRACT ALGEBRA

We begin by defining the j-function:

Definition 1.1. The *j-function* is a function defined on all $\tau \in \mathbb{C}$ (τ can be thought of as an isomorphism class of an elliptic curve, as we'll see later), and we have

$$j(\tau) = 12^3 \cdot \frac{g_2(\tau))^3}{g_2(\tau)^3 - 27g_3(\tau)^2},$$

where

$$g_2(\tau) = 60 \sum_{\substack{m,n\in\mathbb{Z} \\ (m,n)\neq(0,0)}} (m+n\tau)^{-4}$$

and

$$g_3(\tau) = 140 \sum_{\substack{m,n\in\mathbb{Z} \\ (m,n)\neq(0,0)}} (m+n\tau)^{-6}.$$

The *modular discriminant* is defined as

$$\Delta(\tau) = g_2(\tau)^3 - 27g_3(\tau)^2$$

in this context.

The modular discriminant is an infinite sum over certain lattice. We will the modular discriminant more closely later on in this paper. One other way we can write the j-function is

$$j(\tau) = \frac{1}{e^{2\pi i\tau}} + 744 + 196884q + 21493760q^2 + 864299970q^3 + \ldots := \frac{1}{e^{2\pi i\tau}} + \sum_{n=0}^{\infty} c(n)q^n,$$

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where $q = e^{2\pi i\tau}$ [Sch10]. This is often referred to as the q-expansion of $j(\tau)$. Now that we have the basic definition of the j-function, let's go through a bunch of preliminary definitions, lemma, and a corollary[1]:

Definition 1.2. A *simple group* is a nontrivial group with only the trivial group and the group itself as its normal subgroups. There are three infinite classes: cyclic groups of prime order, alternating groups with degree $n \geq 5$, and groups of Lie type. However, the other 26 groups are called *sporadic groups*.

The Monster group, which we will define next, has 19 of the other sporadic groups as either subquotients or subgroups.

Definition 1.3. The *Monster group*, which we will denote $\mathbb{M}$, is the largest simple sporadic group; it has order

$$\begin{aligned}&2^{46} \times 3^{20} \times 5^9 \times 7^6 \times 11^2 \times 13^3 \times 17 \times 19 \times 23 \times 29 \times 31 \times 41 \times 47 \times 59 \times 71\\ &\qquad = 808{,}017{,}424{,}794{,}512{,}875{,}886{,}459{,}904{,}961{,}710{,}757{,}005{,}754{,}368{,}000{,}000{,}000.\end{aligned}$$

This is approximately 8×10^{53}.

The Monster group is one of the most (if not the most) important groups with a connection to the j-function. As we will see later, the j-function is connected to a very special representation of the Monster group, which is called the Monster vertex algebra. Both the Monster group and the Monster vertex algebra are related to the following:

Definition 1.4. We refer to the strange relation between the Monster group and the j-function as *monstrous moonshine*.

We have now seen the basics of Monster groups. Let's turn our attention now to a bit of abstract algebra, which we will be seeing frequently in the coming sections.

Definition 1.5. The multiplicative group of all integer 2×2 matrices with determinant 1,

$$\left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{Z}, \det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = 1 \right\},$$

has a number of applications to the j-function. We denote this group $SL_2(\mathbb{Z})$. [Sch10]

The main thing we will use this multiplicative group for is in theorems and their proofs involving modular functions, which we will look at in the next section. The following lemma is somewhat unrelated to the previous, but relates to meromorphic functions (see footnote 3 on the next page).

Lemma 1.6. *Let $\mathcal{S} := \mathbb{P}^1(\mathbb{C})$ be the Riemann sphere*[2]*. If $f : \mathcal{S} \to \mathbb{C}$ is meromorphic (see footnote 3 on the next page), then $f(z)$ is a rational function.*

Keep some of these theorems etc. in the back of your mind, we'll use them later on. Now that we have the basics, we can look at some properties of the j-function in relation to other modular functions.

[1] I assume some basic knowledge of abstract algebra/groups, but we will see some less trivial definitions throughout the paper.

[2] Riemann sphere, i.e., the complex plane with an added point at infinity.

2. The modular j-function relating to other modular functions

Let's begin by defining what it means for a function to be modular.

Definition 2.1. We call a function $f : H \to \mathbb{C}$ *modular* iff it satisfies these properties:

1. f is meromorphic[3] in the open upper half-plane H.
2. For each matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ (where $a, b, c, d \in \mathbb{Z}$) in the modular group $SL_2(\mathbb{Z})$, we have

$$f\left(\frac{az+b}{cz+d}\right) = f(z).$$

3. The Fourier series for f is of the form

$$f(z) = \sum_{n=-m}^{\infty} a_n e^{2i\pi nz}.$$

This is because the second condition implies periodicity for f, so it is also implied that f has a Fourier series.

One thing that should be fairly obvious by now is that the j-function is a modular function. Naturally, we might wonder about modular functions with certain q-expansions.

Corollary 2.2. *If a modular function has a q-expansion with no negative powers of q (holomorphic at ∞), then it is constant.*

Let's look at the following theorems relating to modular functions and the j-function.

Remark 2.3. Notice that the following two theorems go hand-in-hand–some proofs of the first rely on the second, and some proofs of the second rely on the first. We will prove them in the order below.

Theorem 2.4. *Every modular function (for $SL_2(\mathbb{Z})$) is a rational function in j.* [Sch10]

Proof. The j-function defines the isomorphism from $K(1)$ to $\mathcal{S}$, where $K(1)$ is a modular curve. Let us define $g : K(1) \to \mathbb{C}$ as a modular function for $SL_2(\mathbb{Z})$. Then we have $f = g \circ j^{-1} : \mathcal{S} \to \mathbb{C}$. The latter is meromorphic. From Lemma 1.8, f must be rational, so the theorem is true. ∎

Lemma 2.5. *Every holomorphic modular function for $SL_2(\mathbb{Z})$ is a polynomial in $j(\tau)$.*

Proof. Let $f(\tau)$ be a holomorphic modular function for $SL_2(\mathbb{Z})$. We would like to show that this is a polynomial in $j(\tau)$. Because it is modular and meromorphic in the open upper half-plane H (as required in definition 1.6), we can write its q-expansion:

$$f(\tau) = \sum_{n=-m}^{\infty} a_n q^n.$$

Let us introduce another polynomial g where the powers of the terms of $f(\tau) - g(j(\tau))$ are nonnegative. The latter difference is constant, so we have $f(\tau) = (a + bi) + g(j(\tau))$, and $f(\tau)$ is a polynomial in $j(\tau)$. ∎

[3]A function on an open subset D of the complex plane is called *meromorphic* if it is a function that is holomorphic (differentiable/complex differentiable) on all of D, with the exception of a set of isolated points.

There are more interesting things about the j-function and modular functions in several papers, including Scherer's, but for now that's all we'll look at.

3. Showing that $e^{\pi\sqrt{163}}$ is nearly an integer

One interesting way we can use the j-function is to show that $e^{\pi\sqrt{163}}$ is nearly an integer. More precisely, this number is within nearly 10^{-12} of an integer. We often refer to $e^{\pi\sqrt{163}}$ as *Ramanujan's constant*, or we can use the following notation, which we will be seeing frequently.

Definition 3.1. We often denote e^x ($f(x) = e^x$ is the exponential function) as $\exp x$.

Although we utilize the j-function in the coming proof, we must first look at the following definition.

Definition 3.2. A Heegner number is a square-free positive integer d where the imaginary quadratic field[4] $\mathbb{Q}[\sqrt{-d}]$ has class number[5] 1. There are only 9 Heegner numbers, namely, $1, 2, 3, 7, 11, 19, 43, 67, 163$.

It is worth noting that the Heegner numbers are useful in proving that a wide array of almost-integers are indeed almost-integers. Now we formally present the theorem and proof:

Theorem 3.3. *Ramanujan's constant,* $e^{\pi\sqrt{163}}$*, is an almost-integer.*

Proof. Let us look at the q-expansion of $j(\tau)$:

$$j(\tau) = \frac{1}{q} + 744 + 196884q + 21493760q^2 + \dots.$$

Let $\tau = \frac{1+\sqrt{-163}}{2}$. Then we have $q = -\exp(-\pi^{\sqrt{163}})$; in terms of the first term of the q-expansion, we have $\frac{1}{q} = -\exp(\pi^{\sqrt{163}})$. So we have $j(\frac{1+\sqrt{-163}}{2}) = (-640320)^3$. Let's now write $-e^{\pi^{\sqrt{163}}}$ instead of $-\exp(-\pi^{\sqrt{163}})$, just to see what we have:

$$j\left(\frac{1+\sqrt{-163}}{2}\right) = -e^{\pi^{\sqrt{163}}} + 744 + O(e^{-\pi^{\sqrt{163}}}),$$

where O represents the orthogonal group[6]. Now, say we would like $e^{\pi\sqrt{163}}$ to be within some positive value v of an integer (this number v is sometimes refered to as the *error*). The linear term of v is $-\frac{196884}{e^{\pi\sqrt{163}}}$. We have

$$e^{\pi\sqrt{163}} - v = 196884e^{-\pi\sqrt{163}} + 21493760e^{-2\pi\sqrt{163}} + \dots,$$

and this is approximately 0. So v is approximately $-\frac{-196884}{640320^3+744} = -0.00000000000075$, and this shows that Ramanujan's constant is very close to an integer (about 262537412640768744, so the constant is approximately $262537412640768744 - 0.75 \times 10^{-12}$). ■

[4]The *quadratic field* is an algebraic number field K of degree 2 over $\mathbb{Q}$. We have that $d \to \mathbb{Q}[\sqrt{d}]$ is a bijection from the set of square-free integers to the set of quadratic fields. If d is negative, we call this an *imaginary quadratic field*.

[5]The *(ideal) class group* of an algebraic number field K is the quotient group J_K/P_K (J_K is the group of fractional ideals of the ring of integers of K, and P_K is the subgroup. The *class number* is the cardinality of the ideal class group of its ring of integers.

[6]The *orthogonal group* is a group that is both algebraic and a Lie group (which we will discuss briefly). The orthogonal group is the group of distance-preserving transformations of an n-dimensional Euclidean space that preserve a fixed point.

Now that we have been focusing on the latter theorem for a while, we'll do something different and look at the j-function as it relates to elliptic curves.

4. The j-function and elliptic curves

As we are used to by now, we begin this section with a bunch of definitions (although we can't find very many interesting facts other than these definitions):

Definition 4.1. An *elliptic curve* is an algebraic plane curve (i.e., zero set of two-variable polynomial) given by the equation $y^2 = x^3 + ax + b$, with no cusps (points which move backwards) or self-intersections. Elliptic curves are of genus 1 (meaning that they have 1 "hole").

Elliptic curves are not to be confused with the actual curve of an ellipse; they are called "elliptic" because they have been used in problems involving arc length of an ellipse. In the following definition, we will learn an important term in abstract algebra, which also has a relation to elliptic curves and j-functions, as we'll see in the coming proposition.

Definition 4.2. An *isomorphism* is a map between two similar structures (and it is reversible). *Isomorphic* is the adjective describing this. An *isomorphism class* is a collection of objects which are isomorphic to each other.

Remark 4.3. What exactly is an "object?" It depends, and the term is used widely among many branches of mathematics. In the case of abstract algebra, however, it is likely that an "object" is a group, ring, field, lattice, vector space, etc.

Proposition 4.4. *The j-function is what specifies the isomorphism classes of elliptic curves.*

One way of seeing that the above is true is noticing that τ in $j(\tau)$ represents the isomorphism class. It's natural to wonder if j is actually defined for everywhere in H (upper half-plane). It is, and this is because the modular discriminant (of the Weierstrass elliptic functions), which was defined in the beginning of this paper, is nonzero. If we want to give a better definition of $\Delta(\tau)$, we must first introduce Weierstrass's elliptic function:

Definition 4.5. *Weierstrass's elliptic function* is an elliptic function with periods ω_1 and ω_2 where

$$\phi(z;\omega_1,\omega_2) = \frac{1}{z^2} + \sum_{m\omega_1+n\omega_2\neq 0} \left\{ \frac{1}{(z+m\omega_1+n\omega_2)^2} - \frac{1}{(m\omega_1+n\omega_2)^2} \right\}.$$

This function can also be thought of as the inverse of an elliptic integral. Let

$$u = \int_y^\infty \frac{ds}{\sqrt{4s^3 - g_2 s - g_3}}.$$

Then we have $y = \phi(u)$.

Now let's go back to the definition of $\Delta(\tau)$ in the first definition of the j-function.

Definition 4.6. Let us review what $\Delta(\tau)$ is in the first definition of the j-function:

$$\Delta(\tau) = g_2(\tau)^3 - 27g_3(\tau)^2.$$

This is called the *modular discriminant*, and it is defined to be the quotient by 16 of the discriminant of the right side of the Weierstrass elliptic function written as an integral.

5. Miscellaneous properties of the j-function

We'll just glance at a few interesting properties of the j-function:

Property 5.1. *The fact that* $j\left(\frac{1+\sqrt{-163}}{2}\right) = -640320^3$ *was used to prove that*

$$\frac{1}{\pi} = \frac{12}{640320^{3/2}} \sum_{k=0}^{\infty} \frac{(6k)!(163 \times 3344418k + 13591409}{(3k)!(k!)^3(-640320)^{3k}}.$$

We'll define some more abstract algebra terminology here (the following two definitions are paraphrased from [Pin10]).

Definition 5.2. If K is the root field of polynomial $f(x)$ in $F[x]$, then the group of all the automorphisms (isomorphisms from the object–in this case, the root field–to itself) of K which fix (map to itself) F is called the *Galois group* of $f(x)$. [Pin10]

The Galois group is one of the most famous groups in abstract algebra.

Definition 5.3. Let K be a subfield of some field F. Then F is called an *extension field* of K. [Pin10]

Notice that we can have a Galois extension, essentially the combination of the previous two definitions.

Definition 5.4. An *abelian group* is a group satisfying commutativity.

Now for the second and third properties:

Property 5.5. *The extension field* $\mathbb{Q}[j(\tau), \tau]/\mathbb{Q}(\tau)$ *has an abelian Galois group and is itself abelian.*

Property 5.6. *If* τ *belongs to an imaginary quadratic field with a positive imaginary part, then* $j(\tau)$ *is the root of a monic polynomial with integer coefficients. That's a mouthful, but the latter is analogous to saying that* $j(\tau)$ *is an algebraic integer.*

Now we turn to one of the most fascinating properties of the j-function–its relation to the Monster group.

6. The j-function and monstrous moonshine

Definition 6.1. A *graded ring* is a ring where the additive group is a direct sum of abelian groups G_a such that $G_aG_b \subseteq G_{a+b}$. An algebra (for now, think algebraic structure) A over a ring R is called a *graded algebra* if it is a graded ring.

How, exactly, is the j-function related to monstrous moonshine?

Theorem 6.2. *Assume we have an infinite-dimension graded algebra of the Monster group (as defined earlier). The coefficients of the positive powers of* q *in the* q*-expansion of* $j(\tau)$ *are the dimensions of the graded part of the graded algebra.*

It turns out that this isn't the only beautiful relation between the j-function and the monster group. As always, we need preliminary definitions.

Definition 6.3. Abstractly speaking, the *Jabobi identity* tells us how the order of evaluation (layout of parentheses) will work for a given operation.

Definition 6.4. A *Lie algebra* is a collection of vectors (vector space) $\mathfrak{g}$ having a non-associative operation with an alternating bilinear map satisfying the Jacobi identity.

The following two definitions are closely related due to the Kac-Moody algebra.

Definition 6.5. A *Kac-Moody algebra* is a (infinite-dimensional) Lie algebra which is defined by generators and certain relations (using a generalized Cartan matrix, but that is not so important here). A *generalized Kac-Moody algebra* is also a Lie algebra, and the main difference is that, unlike the regular Kac-Moody algebra, it can have simple imaginary roots.

Now that we know what a Kac-Moody algebra is, we can define the Monster Lie algebra, perhaps one of the most complex and important definitions bringing together the j-function and the monster group.

Definition 6.6. The *Monster Lie algebra* is an infinite-dimensional generalized Kac–Moody algebra. The vector $(1, -1)$ gives this algebra one real simple root. The *Monster vertex algebra* is loosely defined as an algebra related to the monster group, and it was used to prove the connection between the j-function and the monster group.

We will end with an interesting corollary.

Corollary 6.7. *The coefficient c of q^n in the q-expansion of $j(\tau)$ is the c^{th}-dimension of the grade-n part of the Monster vertex algebra.*

References

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