# The Jacobi Triple-Product

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**Abstract**

The Jacobi Triple Product is the elegant identity first discovered and published by Carolus Gustavus Jacobi in his *Fundamenta Nova Theoriae Functionum Ellipticarum* alongside his work on certain elliptic functions. Useful in many areas of number theory, the Jacobi Triple Product has applications in the Theory of Integer Partitions, emerges as special cases in both the Weyl Denominator Formula and the Macdonald Identities, and also provides a generalization of the Euler Pentagonal Number Theorem. The proof provided in this paper is constructed by introducing some generating functions, the properties of which enable the algebraic manipulation that produces the identity.

## 1 A Few Preliminaries

We will begin our derivation of the Jacobi Triple Product formula by studying the behaviour of some generating functions. First, we shall define our two generating functions, and then prove that they satisfy some interesting functional equations.

**Definition 1.1.** We define the generating function $\Phi(z)$ as such:

$$\Phi(z) = \prod_{n=1}^{\infty} \left( \frac{x^{2n-1}}{z^2} + 1 \right) \left( z^2 x^{2n-1} + 1 \right).$$

**Definition 1.2.** We define the second generating function $\Psi(z)$ in terms of $\Phi(z)$:

$$\Psi(z) = \Phi(z) \prod_{n=1}^{\infty} (1 - x^{2n}).$$

Later on, we will equate $\Psi(z)$ to the LHS of the Jacobi Triple Product by using a Laurent series, and use the functional equations we study below to produce the three-term product on the RHS.[1]

**Lemma 1.3.**

$$\Phi(z) = xz^2 \Phi(xz).$$

[1]This derivation closely follows the methods used in [2], which discusses in detail the applications of the Jacobi Triple Product to finding lattice points on spheres.

*Proof.* Through substituting $xz$ into the definition of $\Phi(z)$, we have:

$$\Phi(xz) = \prod_{n=1}^{\infty} \left( \frac{x^{2n+1}}{z^2} + 1 \right) \left( x^{2n+1} z^2 + 1 \right).$$

We proceed by multiplying the second factor in our product by $\frac{1+xz^2}{1+xz^2}$, to yield:

$$\Phi(xz) = \frac{1}{1+xz^2} \prod_{n=1}^{\infty} \left( x^{2n-1} z^2 + 1 \right) \prod_{n=1}^{\infty} \left( \frac{x^{2n+1}}{z^2} + 1 \right)$$

From here, it is possible to bring out the first term of the second product and re-index to 1. Then, we have

$$\begin{aligned}\Phi(xz) &= \frac{1}{1+xz^2} \prod_{n=1}^{\infty} \left( x^{2n-1} z^2 + 1 \right) \cdot \left( 1 + \frac{1}{xz^2} \right) \prod_{n=1}^{\infty} \left( \frac{x^{2n-1}}{z^2} + 1 \right) \\ &= \left( \frac{1}{1+xz^2} \right) \left( 1 + \frac{1}{xz^2} \right) \prod_{n=1}^{\infty} \left( x^{2n-1} z^2 + 1 \right) \left( \frac{x^{2n-1}}{z^2} + 1 \right).\end{aligned}$$

Note that the infinite product is equal to $\Phi(z)$. So, we have

$$\begin{aligned}\Phi(xz) &= \left( \frac{1}{1+xz^2} \right) \left( \frac{1+xz^2}{xz^2} \right) \Phi(z) \\ &= \frac{\Phi(z)}{xz^2}.\end{aligned}$$

So, we find that

$$\Phi(z) = xz^2 \Phi(xz),$$

as desired.

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**Lemma 1.4.** We find that the same property holds for $\Psi(z)$ as well, namely:

$$\Psi(z) = xz^2 \Psi(xz).$$

*Proof.* Again, we substitute $xz$ for $z$ into the definition of the generating function in question, $\Psi(z)$. This yields

$$\Psi(xz) = \Phi(xz) \prod_{n=1}^{\infty} \left( 1 - x^{2n} \right).$$

Now, we can make use of the result from Lemma 1.3, and substitute $\frac{\Phi(z)}{xz^2}$ for $\Phi(xz)$. Then, we have

$$\begin{aligned}\Psi(xz) &= \frac{\Phi(xz)}{xz^2} \prod_{n=1}^{\infty} \left( 1 - x^2 n \right) \\ &= \frac{\Psi(z)}{xz^2}.\end{aligned}$$

Thus,

$$\Psi(z) = xz^2\Psi(xz).$$

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**Lemma 1.5.** There is yet another functional equation that $\Phi$ and $\Psi$ hold in common:

$$\Phi(z) = \Phi\left(\frac{1}{z}\right),$$

*and*

$$\Psi(z) = \Psi\left(\frac{1}{z}\right).$$

*Proof.* The results follow after substituting $x = z^{-2}$ into Lemmas 1.3 and 1.4.

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# 2 Laurent Series

**Definition 2.1.** A *Laurent Series* for a complex-valued function $f(z)$ centered at $z = z_0$ is an infinite series representation of $f(z)$ of the form

$$\sum_{k=1}^{\infty} \frac{a_k}{(z-z_0)^k} + \sum_{k=0}^{\infty} b_k(z-z_0)^k,$$

or more simply:

$$\sum_{k=-\infty}^{\infty} c_k(z-z_0)^k.$$

Unlike the Taylor series, which is an infinite series composed from terms raised to non-negative powers, the terms of Laurent series include negative powers as well. A key difference between Taylor and Laurent series is that while the former represents holomorphic functions defined on a disc centered at $z_0$, the latter is more versatile, being able to represent holomorphic functions defined on an annulus centered at $z_0$. In practice, the Laurent series is used on complex functions when the use of Taylor series is impossible. As a consequence, the class of functions that can be represented by a Laurent series is far more expansive than those compatible with the Taylor series. [2]

For the next step in the derivation of the Jacobi Triple Product, we shall construct a Laurent series expansion for $\Psi(z)$.

**Lemma 2.2.** There exist Laurent[3] coefficients $c_k$ such that $c_k = c_{-k}$, and

$$\Psi(z) = \sum_{k=-\infty}^{\infty} c_k(z^{2k}).$$

[2]For a more extensive look into Laurent Series, visit [1].

[3]We assume here that $\Psi(z)$ is a holomorphic function defined on an annulus. The proof is not included, as it is beyond the scope of this paper.

*Proof.* We begin by noting that $\Psi(z)$ is an even function, which follows after checking that $\Phi(-z) = \Phi(z)$ in 1.1, and using this fact to show that $\Psi(-z) = \Psi(z)$ in 1.2. Since $\Psi(z)$ is an even function, all odd powers will vanish in its Laurent series expansion, thus yielding the series

$$\Psi(z) = \sum_{k=-\infty}^{\infty} c_k(z^{2k}).$$

Now, all that is left to prove is that $c_k = c_{-k}$. This is accomplished after applying Lemma 1.5, with which we can show that

$$\begin{aligned}\Psi(z) &= \sum_{k=-\infty}^{\infty} c_k(z^{2k}) \\ &= \sum_{k=-\infty}^{\infty} c_k(z^{-2k}).\end{aligned}$$

This implies that $c_k = c_{-k}$, completing the proof.

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**Lemma 2.3.** The coefficients of the Laurent series for $\Psi(z)$ satisfy the following condition:

$$c_k = c_{k-1}x^{2k-1} \quad \forall k \in \mathbb{Z}.$$

*Proof.* We showed in Lemma 1.4 that $\Psi(z) = xz^2\Psi(xz)$. Together with the Laurent expansion of $\Psi(z)$, we can write

$$\begin{aligned}\Psi(z) &= \sum_{k=-\infty}^{\infty} c_k(z^{2k}) \\ &= xz^2 \sum_{k=-\infty}^{\infty} c_k\big((xz)^{2k}\big) \\ &= \sum_{k=-\infty}^{\infty} c_k(x^{2k+1}z^{2k+2}).\end{aligned}$$

Then, we can re-index this last sum by setting $k = j - 1$, which yields the sum

$$\Psi(z) = \sum_{j=-\infty}^{\infty} c_{j-1}(x^{2j-1}z^{2j}).$$

Now, we can equate this sum with the sum from Lemma 2.2 and equate the coefficients of the $z^{2k}$ term, after which we have

$$c_k = c_{k-1}x^{2k-1}.$$

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**Lemma 2.4.** The Laurent coefficients of the $\Psi$ generating function satisfy another useful property, namely:

$$c_k = c_0 x^{k^2} \quad \forall k \in \mathbb{Z}$$

*Proof.* We provide a proof by induction. The base case for $k = 0$ is evident. Now, assume that for some $k$, $c_k = c_0 x^{k^2}$. Then, using the result from Lemma 2.3, we have:

$$\begin{aligned} c_{k+1} &= c_k x^{2k+1} \\ &= c_0 x^{k^2+2k+1} \\ &= c_0 x^{(k+1)^2}, \end{aligned}$$

Thus completing the proof.

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**Lemma 2.5.** We note that the first Laurent coefficient for $\Psi(z)$ has the following value:

$$c_0 = 1$$

*Proof.* To prove that $c_0 = 1$, we will equate two different calculations of $\Psi(1)$. The first method will be to substitute $z = 1$ into the definition for $\Psi(z)$ given in 1.2, which requires the calculation of $\Phi(1)$. So, by the definition of $\Phi(z)$ given in 1.1, we have

$$\Phi(1) = \prod_{n=1}^{\infty} (1 + x^{2n-1})^2.$$

Substituting this into the expression for $\Psi(1)$, we get

$$\begin{aligned} \Psi(1) &= \Phi(1) \prod_{n=1}^{\infty} \left(1 - x^{2n}\right) \\ &= \prod_{n=1}^{\infty} \left(1 + x^{2n-1}\right)^2 \left(1 - x^{2n}\right). \end{aligned}$$

Alternatively, we could calculate $\Psi(1)$ by substituting $z = 1$ into 2.2, which yields

$$\Psi(1) = \sum_{k=-\infty}^{\infty} c_k.$$

Then, we can utilize the formula for $c_k$ that we derived in 2.4, after which we would have

$$\Psi(1) = c_0 \sum_{k=-\infty}^{\infty} x^{k^2}.$$

Together with the other equation for $\Psi(1)$ that we found earlier, it is implied that there is no coefficient on the sum above, meaning that $c_0$ must be equal to 1.

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# 3 Proof of the Jacobi Triple Product

To begin our last steps towards proving the Jacobi Triple Product, we will have to rewrite $\Psi(z)$. Combining the expression for the Laurent coefficient $c_k$ that we found in 2.4, the fact that $c_0 = 1$, and the Laurent series expansion that we found in 2.2, we have

$$\Psi(z) = \sum_{k=-\infty}^{\infty} x^{k^2} z^{2k}.$$

By substituting the definition of $\Phi(z)$ from 1.1 into the definition for $\Psi(z)$ from 1.2, we have

$$\Psi(z) = \prod_{n=1}^{\infty} \left(1 + x^{2n-1}z^2\right)\left(1 + \frac{x^{2n-1}}{z^2}\right)\left(1 - x^{2n}\right).$$

Then, all that is left to do is to equate this formula for $\Psi(z)$ with the one we derived earlier, thus producing the Jacobi Triple Product formula:

$$\sum_{k=-\infty}^{\infty} x^{k^2} z^{2k} = \prod_{n=1}^{\infty} \left(1 + x^{2n-1}z^2\right)\left(1 + \frac{x^{2n-1}}{z^2}\right)\left(1 - x^{2n}\right). \tag{1}$$

# 4 Euler's Pentagonal Number Theorem

The *Euler Pentagonal Number Theorem* is a special case of the Jacobi Triple Product that equates the infinite product in the *Euler Function* given by $E(z) = \prod_{n=1}^{\infty}(1 - z^n)$ (not to be confused with the Euler *Totient* function) to its series representation. Before we derive the Pentagonal Number Theorem, we shall look at a few related special cases of the Jacobi Triple Product.

**Example 1.** We can perform some simple substitutions upon the Jacobi Triple Product to obtain a generating function for square numbers. If we let $z$ equal 1 in (1), we get:

$$\sum_{k=-\infty}^{\infty} x^{k^2} = \prod_{n=1}^{\infty} \left(1 + x^{2n-1}\right)^2\left(1 - x^{2n}\right).$$

Similarly, we can perform different substitutions to obtain generating functions for other types of numbers, like triangular and pentagonal numbers.

**Example 2.** Performing the substitution $z = x$ followed by $x = \sqrt{x}$ in the Jacobi Triple Product from (1) yields

$$\sum_{k=-\infty}^{\infty} x^{\frac{k(k+1)}{2}} = \prod_{n=1}^{\infty} 2\left(1 + x^n\right)^2\left(1 - x^n\right).$$

This is the generating function for the triangular numbers; the exponent of the summand is $\frac{k(k+1)}{2}$, which follow the pattern $1, 3, 6, 10, \ldots$.

**Example 3.** Euler's Pentagonal Number Theorem is derived in a similar fashion; we perform the substitutions $x = x^{\frac{3}{2}}$ and then $z = -\sqrt{x}$ on the Jacobi Triple Product to get:

$$\sum_{k=-\infty}^{\infty} (-1)^k x^{\frac{k(3k+1)}{2}} = \prod_{n=1}^{\infty} \left(1 - x^{3n-1}\right)\left(1 - x^{3n-2}\right)\left(1 - x^{3n}\right)$$
$$= \prod_{n=1}^{\infty} \left(1 - x^{n}\right),$$

thus yielding the desired result.

# References


[1] V. V. Datar. Lecture-22 : Laurent series. Course notes, 2016.

[2] R. Zaman. The jacobi triple product. Unpublished manuscript, 2011.