# INTRODUCTION THE J-INVARIANT

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ABSTRACT. In this paper, we study the $j$-function, an important tool in complex analysis and modular theory. It defines a crucial invariance for elliptic curves, and also, as we prove, serves as a bijection between $\mathrm{SL}_2(\mathbb{Z})\backslash\mathcal{H}$ and $\mathbb{C}$. In order to define this function, we go over the basics of representation theory, and then explain more about lattices and the functions such as the Eisenstein series and the Weierstrass elliptic function. Lastly, we move on to modular functions, and set up the foreknowledge required to define the $j$-function as a modular function itself. We conclude with some interesting properties about the said function and a proof of aforementioned bijection.

## 1. Representations

We start with the definition of a representation:

**Definition 1.1.** A *representation* of a group $G$ over a field $F$ is a vector space $V$ together with a group homomorphism $\phi$ that sends $G$ to $\mathrm{GL}(V, \mathbb{C})$, the general linear group of $V$ over $\mathbb{C}$. We denote this representation as $(V, \phi)$

Note that a group homomorphism is a map between two groups that preserves the group operation. Because representations are essentially vector spaces, we can consider the representations of their subspaces:

**Definition 1.2.** If $W$ is a subspace of $V$ and $\phi(g)(w) \in W \quad \forall g \in G, w \in W$, then we call $W$ a *subrepresentation* of $V$.

We can also combine representations together by taking the *direct sum*:

**Definition 1.3.** The *direct sum* of two representations $(U, \rho)$ and $(V, \phi)$ of a group $G$ is the representation $(U \oplus V, \rho\phi)$.

A representation is called *irreducible* if it cannot be expressed as the direct sum of its subrepresentations, and *reducible* if it can. In fact, any representation can be expressed as a unique direct sum of irreducible representations; see [FH91] for a proof.

**Definition 1.4.** Let $g \in G$ and $(V, \phi)$ be a representation of $G$. Then the *character* of $g$ with respect to $V$, usually denoted as $\chi_V(g)$, is the trace of $\phi(g)$.

*Remark* 1.5. If $V$ has dimension $n$, then $\chi_V(e) = n$.

Characters have the interesting property that they are constant over conjugacy classes, i.e. $\chi_V(hgh^{-1}) = \chi_V(g)$ for $g, h \in G$. [FH91]

## 2. Lattices

After establishing a basic understanding of representation theory, we move on to *lattices*:

**Definition 2.1.** A *lattice* $L$ is an additive subgroup of $\mathbb{C}$ generated by two nonzero complex numbers $\omega_1, \omega_2$ such that $\Im(\frac{\omega_1}{\omega_2}) \neq 0$.

This last condition is necessary to ensure $\omega_1$ and $\omega_2$ are linearly independent. One common example of a lattice is $\mathbb{Z}[i]$, the lattice generated by 1 and $i$.

**Definition 2.2.** Two lattices $L_1, L_2$ are said to be *homothetic* if there exists $\lambda \in \mathbb{C}$ such that $L_1 = \lambda L_2$.

Basically, two lattices are homothetic if the ratios between their generators are equal. We call the equivalence class generated by a homothety a *homothety class*.

We now introduce the *Eisenstein series* along with the *Weierstrass elliptic function*:

**Definition 2.3.** Let $n \geq 3$ and $L$ be a lattice. We define the *Eisenstein series* of order $n$ to be

$$E_n(L) = \sum_{\omega \in L \setminus \{0\}} \frac{1}{\omega^n}.$$

**Proposition 2.4.** *$E_n(L)$ converges absolutely for real $n \geq 3$ and lattice $L$.*

*Proof.* Let $\omega_1, \omega_2 \in \mathbb{C}$ be the generators of $L$. We will show that

$$\sum_{\omega \in L \setminus \{0\}} \left| \frac{1}{\omega^n} \right| = \sum_{\substack{x,y \in \mathbb{Z} \\ (x,y) \neq (0,0)}} \frac{1}{|x\omega_1 + y\omega_2|^n}$$

converges. Now we define $M \in \mathbb{R}$ as the minimum value of $|\omega_1 \cos\alpha + \omega_2 \sin\alpha|$ for $\alpha \in \mathbb{R}$. Then setting $\alpha = \cos^{-1}\left(\frac{x}{\sqrt{x^2+y^2}}\right)$ gives

$$|\omega_1 \cos\alpha + \omega_2 \sin\alpha| = \left| \frac{x}{\sqrt{x^2+y^2}}\omega_1 + \frac{y}{\sqrt{x^2+y^2}}\omega_2 \right| \geq M,$$

or $|x\omega_1 + y\omega_2| \geq M\sqrt{x^2+y^2}$. Substituting, our summation satisfies the inequality

$$\sum_{\omega \in L \setminus \{0\}} \left| \frac{1}{\omega^n} \right| \leq \frac{1}{M^n} \sum_{\substack{x,y \in \mathbb{Z} \\ (x,y) \neq (0,0)}} \frac{1}{(x^2+y^2)^{\frac{n}{2}}} \leq \frac{1}{M^n} \iint_{x^2+y^2 \geq 1} \frac{1}{(x^2+y^2)^{\frac{n}{2}}}.$$

Thus it suffices to show that this integral converges, which immediately follows from a polar substitution. □

**Definition 2.5.** For $z \in \mathbb{C}$ and lattice $L$, we define the *Weierstrass elliptic function* as

$$\wp(z; L) = \frac{1}{z^2} + \sum_{\omega \in L \setminus \{0\}} \left( \frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} \right).$$

*Remark* 2.6. The Weierstrass elliptic function is even, i.e. $\wp(z; L) = \wp(-z; L)$.

Note that $\omega \in L \setminus \{0\}$ implies $-\omega \in L \setminus \{0\}$. The Weierstrass elliptic function is an example of an *elliptic function*, as a meromorphic function satisfying $\wp(z + \omega) = \wp(z)\ \forall \omega \in L$. We will briefly touch upon *elliptic functions* in Chapter 3.

The Weierstrass elliptic function satisfies the following properties, of which we will prove a few relevant to this paper:

**Proposition 2.7.** *The set of singularities of $\wp(z;L)$ is equivalent to the set of lattice points in L [Cox89].*

**Proposition 2.8.** *The Laurent expansion of $\wp(z;L)$ at 0 is*

$$\wp(z;L) = \frac{1}{z^2} + \sum_{n=1}^{\infty} a_n z^{2n},$$

*where*

$$a_n = (2n+1) \sum_{\omega \in L \setminus \{0\}} \frac{1}{\omega^{2n+2}}.$$

*Proof.* Let $S$ be a summation given by

$$S = \frac{1}{\omega^2} \sum_{k=1}^{\infty} (k+1) \left(\frac{z}{\omega}\right)^k.$$

This is essentially a geometric sum, so we solve to get

$$\left(\frac{z}{\omega}\right)^{-1} S = \frac{1}{\omega^2} \sum_{k=0}^{\infty} (k+2) \left(\frac{z}{\omega}\right)^k$$

$$\left(1 - \frac{\omega}{z}\right) S = -\frac{2}{\omega^2} - \frac{1}{\omega^2} \sum_{k=1}^{\infty} \left(\frac{z}{\omega}\right)^k = -\frac{2}{\omega^2} - \frac{1}{\omega^2} \cdot \frac{\frac{z}{\omega}}{1 - \frac{z}{\omega}}$$

$$S = -\frac{2z}{\omega^2(z-\omega)} + \frac{z^2}{\omega^2(z-\omega)^2} = \frac{2\omega z - z^2}{\omega^2(z-\omega)^2}$$

$$S = \frac{1}{(z-\omega)^2} - \frac{1}{\omega^2}.$$

Thus, the Weierstrass elliptic function can be expressed as

$$\wp(z;L) = \frac{1}{z^2} + \sum_{\omega \in L \setminus \{0\}} \left( \frac{1}{\omega^2} \sum_{k=1}^{\infty} (k+1) \left(\frac{z}{\omega}\right)^k \right).$$

Using [RS15], we simplify to get

$$\wp(z;L) = \frac{1}{z^2} + \sum_{k=1}^{\infty} b_k z^k,$$

where

$$b_k = (k+1) \sum_{\omega \in L \setminus \{0\}} \frac{1}{\omega^{k+2}}.$$

But observe that if $k$ is odd, then

$$b_k = \frac{1}{2}(k+1) \sum_{\omega \in L \setminus \{0\}} \left(\frac{1}{\omega^{k+2}}\right) + \frac{1}{2}(k+1) \sum_{-\omega \in L \setminus \{0\}} \left(\frac{1}{(-\omega)^{k+2}}\right) = 0.$$

Thus, we can eliminate all the odd terms and let $a_k = b_{2k}$, giving us our desired result. ☐

Expanding out the first few terms of the Laurent expansion, we define

$$g_2 = 60E_4(L) = 60 \sum_{\omega \in L \setminus \{0\}} \frac{1}{\omega^4}$$

$$g_3 = 140E_6(L) = 140 \sum_{\omega \in L \setminus \{0\}} \frac{1}{\omega^6}$$

so that

$$\wp(z; L) = \frac{1}{z^2} + \frac{g_2}{20}z^2 + \frac{g_3}{28}z^4 + \mathcal{O}(z^6).$$

Observe that $a_n$ can actually be written as a polynomial in $g_2, g_3$. Now, we define the *discriminant* of a lattice $L$, a crucial part of the $j$-function.

**Definition 2.9.** The *discriminant* of lattice $L$ is given by

$$\Delta(L) = g_2 L^3 - 27g_3 L^2.$$

Interestingly, $\Delta(L)$ is also the discriminant of the polynomial $p(x) = 4x^3 - g_2 x - g_3$ [Cox89]. We also have the following property:

**Proposition 2.10.** *The Weierstrass elliptic function satisfies the differential equation*

$$\wp'(z)^2 = 4\wp(z)^3 - g_2\wp(z) - g_3.$$

*Proof.* Observe the polynomial $f(z) = \wp'(z)^2 - 4\wp(z)^3 + g_2\wp(z) + g_3$. Using the Laurent expansion of the elliptic function, we get

$$f(z) = \mathcal{O}(z^2),$$

so $f$ has no poles at 0. Because $\wp(z)$ is meromorphic and doubly-periodic, it is an *elliptic function*, which we will define later on in the paper. Clearly the powers of an elliptic function are elliptic, and the derivative is also (which we will refer to later). But all elliptic functions with poles are bounded, so by Liouville's Theorem $f$ is constant, and taking $z = 0$ gives $f(z) = 0$, as desired. ☐

**Corollary 2.11.** *The discriminant of a lattice is always nonzero.*

By [Apo90], the roots of $p(x)$ (defined above) are distinct. Then this corollary follows immediately from the fact that the discriminant of the polynomial is the product of the squares of the differences between its roots.

## 3. Modular Functions

We denote the upper-half plane as $\mathcal{H} = \{z \in \mathbb{C} \mid \Im(z) > 0\}$. In the context of modular functions, the group of invertible $2 \times 2$ matrices, $\mathrm{SL}_2(\mathbb{Z})$, acts on $\mathcal{H}$ through a linear fractional transformation. Specifically, for $\tau \in \mathcal{H}$ and $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$, we have

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \tau = \frac{a\tau + b}{c\tau + d}.$$

*Remark* 3.1. $\mathrm{SL}_2(\mathbb{Z})$ is commonly referred to as the modular group $\Gamma$.

**Definition 3.2.** Two points $\tau_1, \tau_2 \in \mathcal{H}$ are considered $\mathrm{SL}_2(\mathbb{Z})$-*equivalent* if there exists $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ such that $\tau_1 = \gamma \cdot \tau_2$.

With this, we can construct a region in $\mathcal{H}$ that represents all $\mathrm{SL}_2(\mathbb{Z})$ equivalence classes:

**Definition 3.3.** The *fundamental region* of $\Gamma$ is defined by

$$\mathcal{D} = \left\{ z \in \mathcal{H} \,\middle|\, |z| > 1,\, -\frac{1}{2} \leq \Re(z) < \frac{1}{2} \right\} \cup \left\{ z \in \mathcal{H} \,\middle|\, |z| = 1,\, -\frac{1}{2} \leq \Re(z) \leq 0 \right\}.$$

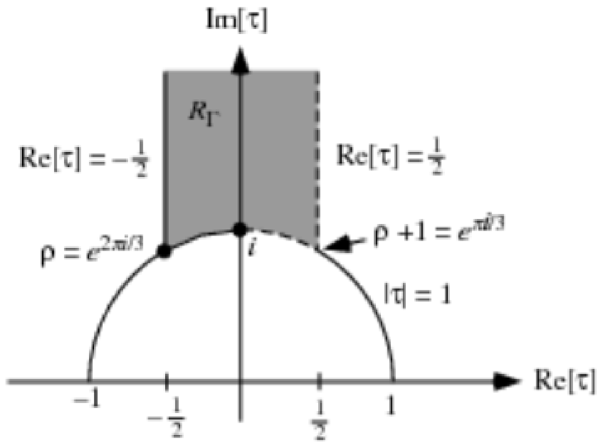


**Figure 1.** The fundamental domain, here denoted $R_\Gamma$ [BB87]

**Proposition 3.4.** $\mathcal{D}$ *satisfies the following properties:*

(1) *Any two distinct points in* $F$ *are not* $\mathrm{SL}_2(\mathbb{Z})$*-equivalent.*
(2) *Every point in* $\mathcal{H}$ *is* $\mathrm{SL}_2(\mathbb{Z})$*-equivalent to a point on* $\mathcal{D}$*.*

This is not too difficult given that the generators of $\mathrm{SL}_2(\mathbb{Z})$ are

$$S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \text{and} \quad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

corresponding to the maps $S\colon z \mapsto -\frac{1}{z}$ and $T\colon z \mapsto z+1$.

**Definition 3.5.** An *elliptic function* $f$ is a meromorphic function that is periodic in two ways. That is, there exists nonzero $\omega_1, \omega_2 \in \mathbb{C}$, denoted as *periods*, with $\Im(\frac{\omega_1}{\omega_2}) \neq 0$ such that

$$f(z) = f(z+\omega) \quad \text{and} \quad f(z) = f(z+\omega_2) \quad \forall z \in \mathbb{C}\,.$$

Clearly the Weierstrass elliptic function satisfies these properties; its periods are just the generators of the lattice. Elliptic functions satisfy the interesting property that taking the derivative results in another elliptic function.

**Definition 3.6.** A function $f\colon\, \mathcal{H} \to \mathbb{C}$ is called a *modular function* if

(1) $f$ is meromorphic in $\mathcal{H}$.
(2) $f(\gamma \cdot \tau) = \tau$ for any $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ and $\tau \in \mathcal{H}$.
(3) The Laurent series of $f$ can be expressed as

$$f(\tau) = \sum_{n=-m}^{\infty} a(n) e^{2\pi i n \tau},$$

called the *q-expansion* of $f$.

Modular functions are essentially *modular forms* of weight 0. However, the subject of modular forms is quite difficult to comprehend, so we will only state the interesting property that holomorphic modular forms of weight $k$ form a finite dimensional vector space $M_k$. One particular linear operator on modular functions is the *Hecke operator*:

**Definition 3.7.** The $n$th *Hecke operator* $T_n$ on the modular function $f$ is defined by

$$T_n f(\tau) = \sum_{d|n} \sum_{a=0}^{d-1} f\left(\frac{n\tau + ad}{d^2}\right).$$

**Proposition 3.8.** *$T_n f$ has the Fourier expansion*

$$T_n f(\tau) = \sum_{m=0}^{\infty} e^{2\pi i m\tau} \left( \sum_{d|\gcd(m,n)} \frac{1}{d} \cdot a\left(\frac{mn}{d^2}\right) \right).$$

## 4. The $j$-invariant

We finally have enough material to construct the $j$-function. Firstly, we define the $j$-*invariant*, which also gives invariance in respect to elliptic curves:

**Definition 4.1.** The *$j$-invariant* of a lattice $L$ is defined as

$$j(L) = 1728 \cdot \frac{g_2(L)^3}{g_2(L)^3 - 27g_3(L)^2},$$

where $g_2, g_3$ are the previously defined coefficients in the Laurent expansion of $\wp(z; L)$.

Observe that $g_2(L)^3 - 27g_3(L)^2 = \Delta(L)$, and by Corollary 2.11 it is never zero. Phew! Now, the *$j$-function* is essentially a $j$-invariant that takes in a complex number instead of a lattice.

**Definition 4.2.** For some $\tau \in \mathcal{H}$, we define the *$j$-function* to be a complex-valued function that sends $\tau$ to the $j$-invariant of the lattice generated by $[1, \tau]$. Modifying $g_2(\tau) = g_2([1, \tau])$ and $g_3(\tau) = g_3([1, \tau])$ gives the equation

$$j(\tau) = 1728 \cdot \frac{g_2(\tau)^3}{g_2(\tau)^3 - 27g_3(\tau)^2}.$$

The $j$-function satisfies various theorems, several of which are listed below:

**Theorem 4.3.** *$j(\tau)$ is holomorphic on $\mathcal{H}$ [Cox89].*

**Theorem 4.4.** *Lattices $L$ and $L'$ are homothetic if and only if $j(L) = j(L')$.*

*Proof.* Suppose $L' = \lambda L$ for some nonzero $\lambda \in \mathbb{C}$. Then we obtain

$$g_2(L') = 60 \sum_{\omega \in L \setminus \{0\}} \frac{1}{(\lambda\omega)^4} = \frac{1}{\lambda^4} g_2(L)$$

$$g_3(L') = 140 \sum_{\omega \in L \setminus \{0\}} \frac{1}{(\lambda\omega)^6} = \frac{1}{\lambda^6} g_3(L)$$

and the factor of $\lambda^{-12}$ cancels, giving $j(L') = j(L)$.

Now we prove the reverse; we are given that $j(L') = j(L)$. Suppose there exists $\lambda \in \mathbb{C}$ with $g_2(L') = g_2(\lambda L)$ and $g_3(L') = g_3(\lambda L)$. Then using the Weierstrass elliptic function gives

$$\wp(z; L') = \frac{1}{z^2} + \sum_{n=1}^{\infty} f(g_2(L'), g_3(L'))z^{2n} = \sum_{n=1}^{\infty} f(g_2(\lambda L), g_3(\lambda L))z^{2n} = \wp(z; \lambda L).$$

Thus $\wp(z; L')$ and $\wp(z; \lambda L)$ must have the same Laurent expansion at 0, so $\wp(z; L') = \wp(z; \lambda L)$ on a neighborhood $U$ about 0. Also note that $\wp(z; L')$ and $\wp(z; \lambda L)$ are analytic on $\Omega := \mathbb{C} \setminus (L' \cup \lambda L)$, and that the set

$$\{z \in \Omega \colon \wp(z; L') = \wp(z; \lambda L)\}$$

has a limit point in $U \cap \Omega$. Thus $\wp(z; L') = \wp(z; \lambda L)$ for all $z \in \Omega$, implying that they have the same poles. Then we are done by Proposition 2.7. Now all we need to do is show that $\lambda$ exists. Notice that $g_2(L')$ and $g_3(L')$ cannot both be zero, so then depending on which one is nonzero, let

$$\lambda = \sqrt[4]{\frac{g_2(L)}{g_2(L')}} \quad \text{or} \quad \lambda = \sqrt[6]{\frac{g_3(L)}{g_3(L')}},$$

and one can show by simple substitution into $j(L') = j(L)$ that the statement holds. □

**Theorem 4.5.** *$\tau_1, \tau_2 \in \mathcal{H}$ are $\mathrm{SL}_2(\mathbb{Z})$-equivalent if and only if $j(\tau_1) = j(\tau_2)$.*

The proof of this is similar to the proof for lattices.

**Proposition 4.6.** *The $j$-function diverges; in other words,*

$$\lim_{\Im(\tau)\to\infty} j(\tau) = \infty.$$

*Proof.* Let $\tau = a + bi$, and consider the value of

$$g_2(\tau) = 60 \sum_{\omega\in[1,\tau]} \frac{1}{\omega^4} = 120 \sum_{m=1}^{\infty} \frac{1}{m^4} + 60 \sum_{m} \sum_{n\neq 0} \frac{1}{(m + n\tau)^4}.$$

But as $n \neq 0$, we have

$$\lim_{\Im(\tau)\to\infty} \frac{1}{(m + n\tau)^4} = \lim_{b\to\infty} \frac{1}{(m + na + nbi)^4} = 0,$$

immediately implying that

$$\lim_{b\to\infty} g_2(a + bi) = 120 \sum_{m=1}^{\infty} \frac{1}{m^4} = \frac{4}{3}\pi^4.$$

Similarly, we have

$$\lim_{b\to\infty} g_3(a + bi) = 280 \sum_{m=1}^{\infty} \frac{1}{m^6} = \frac{8}{27}\pi^6.$$

Thus, we substitute these values in for the $j$-function to get

$$\lim_{b\to\infty} j(a + bi) = 1728 \cdot \frac{\lim\limits_{b\to\infty} g_2(a + bi)^3}{\lim\limits_{b\to\infty} [g_2(a + bi)^3 - 27g_3(a + bi)^2]} = 1728 \cdot \frac{\frac{64}{27}\pi^{12}}{\frac{64}{27}\pi^{12} - 27(\frac{64}{729}\pi^{12})} = \infty$$

□

An immediate consequence of this proposition is that $j(\tau)$ has a pole at $i\infty$, implying it is meromorphic. Along with Theorem 4.5, we obtain the following corollary:

**Corollary 4.7.** *The $j$-function is a modular function for* $\mathrm{SL}_2(\mathbb{Z})$.

**Theorem 4.8.** *The $j$-function is a bijection between* $\mathrm{SL}_2(\mathbb{Z})\backslash\mathcal{H}$ *and* $\mathbb{C}$.

*Proof.* It suffices to prove surjectivity, as Theorem 4.5 implies injectivity. Let $j(\tau_k)$ be a sequence in $\mathbb{C}$ that converges to some $w \in \mathbb{C}$. Due to Theorem 4.5, we can restrict all $\tau_k$ to the fundamental domain $\mathcal{D}$, or that $|\Re(\tau_k)| \leq \frac{1}{2}$ and $|\Im(\tau_k)| \geq \sqrt{3}2$. If $\Im(\tau_k)$ is unbounded, then because of 4.6 $j(\tau_k)$ must contain a subsequence that converges to $\infty$, a contradiction. Thus $\Im(\tau_k)$ is bounded, and $\tau_k$ lies in a compact subspace of $\mathcal{H}$. But then this implies that there exists a subsequence of $\tau_k$ converging to some $\tau' \in \mathcal{H}$. As $j(\tau)$ is continuous and $j(\tau_k)$ converges to $w$, we must have that $j(\tau') = w$, implying that the $w$ lies inside $j(\mathcal{H})$. Therefore, the image of $j(\tau)$ must be a closed set in $\mathbb{C}$. However, we also know that $j(\tau)$ is holomorphic on $\mathcal{H}$ and that $j(\tau)$ is nonconstant. Then by the open mapping theorem, the image of $j(\tau)$ must be an open set. The only set in $\mathbb{C}$ that is clopen is $\mathbb{C}$ itself, so $j(\tau)$ is surjective. ☐

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