The Flint Hills Series

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1 Introduction

Here's the Flint Hills series:

$$\sum_{n=1}^{\infty} \frac{1}{n^3 \sin^2(n)}$$

It's an open problem whether this series converges. Since convergence is related to how often $\sin(n)$ has small values, it's also related to the question of how well π can be approximated by rational numbers.

2 But I Thought π Was Rational

For a number x, the irrationality measure $\mu(x)$ is the smallest (i.e., greatest lower bound of such) m such that

$$\left| x - \frac{p}{q} \right| < \frac{1}{q^m}$$

holds for only finitely many p and q. For instance, let x be rational, say $x = \frac{r}{s}$. Then

$$\left| \frac{r}{s} - \frac{p}{q} \right| = \left| \frac{rq - ps}{qs} \right| \geq \frac{1}{qs}$$

Therefore, for any $\epsilon > 0$, the inequality

$$\left| \frac{r}{s} - \frac{p}{q} \right| < \frac{1}{q^{1+\epsilon}}$$

can only hold for finitely many pairs p, q (it stops working when $q^\epsilon > s$). Since the greatest lower bound of the exponent is 1, the irrationality measure of any rational number is 1. Next are the algebraic numbers.

Definition 2.1 (Algebraic Number)**.** *A number α which is the root of a degree d polynomial with integer coefficients, but not a root of any lower-degree polynomial, is an algebraic number of degree d.*

The first big result regarding irrationality of algebraic numbers is Liouville's Theorem.

Theorem 2.1 (Liouville's Approximation Theorem)**.** *For an algebraic number α of degree d and a rational approximation $\frac{p}{q}$ to α, we have*

$$\left| \alpha - \frac{p}{q} \right| > \frac{C}{q^d}$$

where C is a constant.

Proof. If α is an algebraic number of degree d, then there is a polynomial $f(x) = c_0 + c_1 x + \cdots + c_d x^d$ with α as a root. Note that

$$\left| f\left(\frac{p}{q}\right) \right| = \left| \frac{c_0 q^d + c_1 p^d q^{d-1} + \cdots + c_d p^d}{q^d} \right| \geq \frac{1}{q^d}$$

since the numerator is an integer and $f\left(\frac{p}{q}\right) = 0$ would mean that f is not of minimal degree. We can apply the Mean Value Theorem to see that there must be some b such that

$$f'(b) = \frac{f(\alpha) - f(\frac{p}{q})}{\alpha - \frac{p}{q}}$$

Therefore,

$$\left|\alpha - \frac{p}{q}\right| = \left|\frac{f(\alpha) - f(\frac{p}{q})}{f'(b)}\right| \geq \left|\frac{\frac{1}{q^d}}{f'(b)}\right| = \frac{|f'(b)^{-1}|}{q^d}$$

Let ϵ be such that $(\alpha-\epsilon, \alpha+\epsilon)$ contains no points where f' is zero. Then $C = \min(f'(x)^{-1})$ for $x \in (\alpha-\epsilon, \alpha+\epsilon)$ satisfies the inequality

$$\left|\alpha - \frac{p}{q}\right| > \frac{C}{q^d}$$

for all rational $\frac{p}{q}$ in the interval $(\alpha - \epsilon, \alpha + \epsilon)$, as desired. □

It turns out Liouville's Theorem is relatively weak. The strongest result is

Theorem 2.2 (Thue-Siegel-Roth Theorem)**.** *For an algebraic number α, integral p and q, and $\epsilon > 0$, the inequality*

$$\left|\alpha - \frac{p}{q}\right| < \frac{1}{q^{2+\epsilon}}$$

has only finitely many solutions.

This is just a little too involved for us to show here; Roth got a Fields medal for proving it. The Thue-Siegel-Roth Theorem implies that $\mu(x) = 2$ for any algebraic x. We also know that $\mu(x) \geq 2$ for transcendental x. While it is known that e has irrationality measure of 2, and in fact it has been shown that almost all real numbers have irrationality measure of 2, the irrationality measure of π is not known. All we can say for now is that $2 \leq \mu(\pi) \leq 7.103\ldots$, but if it turns out the Flint Hills Series converges we can shrink that bound even smaller.

3 The Flint Hills Series

This section will show a relationship between the convergence of Flint Hills-like series and $\mu(\pi)$. About 90% of the proof is borrowed from Alekseyev's paper on the topic [1] in which he deals with more general series of the form

$$\sum_{n=1}^{\infty} \frac{1}{n^u |\sin^v(n)|}$$

The first thing we'll talk about is $\mu(\pi)$. By the definition of $\mu(\pi)$, we know that for any $\epsilon > 0$,

$$|\pi - \frac{p}{q}| < \frac{1}{q^{\mu(\pi)+\epsilon}}$$

holds only finitely many times. Before we do anything else, let's have a lemma.

Lemma 3.1. *For all x with $0 < x < \frac{\pi}{2}$, we have $\sin x > \frac{2}{\pi}x$.*

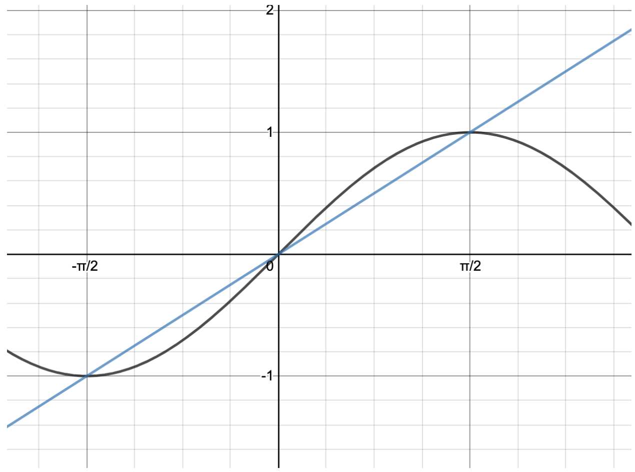

Figure 1: "Proof by Desmos"

Let's put this to use. First, note that there is always some integral m such that $0 < |n - m\pi| < \frac{\pi}{2}$. Since we must also have $\sin n = \sin(n - m\pi)$ for any integral m, we can say that

$$\sin|n - m\pi| = |\sin(n - m\pi)| = |\sin n| > \frac{2}{\pi}|n - m\pi|$$

$$|\sin n| > \frac{2m}{\pi}\left|\pi - \frac{n}{m}\right|$$

This looks like the irrationality inequality for π. By definition of $\mu(\pi)$, we know that

$$\left|\pi - \frac{n}{m}\right| < \frac{1}{m^{\mu(\pi)+\epsilon}}$$

for only finitely many coprime m and n. Therefore, for large enough m and n, we must have

$$\left|\pi - \frac{n}{m}\right| > \frac{1}{m^{\mu(\pi)+\epsilon}}$$

which means

$$|\sin n| > \frac{2}{\pi m^{\mu(\pi)-1+\epsilon}}$$

Since the ratio $\frac{n}{m}$ goes to π as n increases, we can substitute n for m and absorb all our constants into a constant factor:

$$|\sin n| > C\frac{2}{\pi n^{\mu(\pi)-1+\epsilon}}$$

$$|\sin n| > C\frac{1}{n^{\mu(\pi)-1+\epsilon}}$$

We're getting very close to a Flint Hills Series term. A few more manipulations will get us all the way there:

$$\frac{1}{|\sin n|} < C\frac{1}{n^{-\mu(\pi)+1-\epsilon}}$$

$$\frac{1}{|\sin^v n|} < C\frac{1}{n^{-v\mu(\pi)+v-\epsilon}}$$

$$\frac{1}{n^u|\sin^v n|} < C\frac{1}{n^{u-v\mu(\pi)+v-\epsilon}}$$

So, we have

$$\frac{1}{n^u|\sin^v(n)|} = O\left(\frac{1}{n^{u-(\mu(\pi)-1)v-\epsilon}}\right)$$

for large enough n. Observe that the sequence $\frac{1}{n^u|\sin^v n|}$ converges to zero only when $u-(\mu(\pi)-1)v-\epsilon > 0$. So, convergence of $\sum_{n=1}^{\infty}\frac{1}{n^u|\sin^v(n)|}$ implies $\mu(\pi) < 1+\frac{u}{v}$ (however, note that the converse is not necessarily true). We also conclude that $\mu(\pi) > 1+\frac{u}{v}$ implies divergence of both the sequence $\frac{1}{n^u|\sin^v n|}$ and the series $\sum_{n=1}^{\infty}\frac{1}{n^u|\sin^v(n)|}$. For the regular Flint Hills Series, $u = 3$ and $v = 2$, so convergence would imply $\mu(\pi) \leq 2.5$. For context, the most recent upper bound on $\mu(\pi)$ is $7.103205\ldots$ [4].

4 Continued Fraction Approximations

To calculate the continued fraction $[a_0; a_1, a_2 \ldots]$ for a number x, we take $a_0 = \lfloor x \rfloor$, then $a_1 = \lfloor \frac{1}{x-\lfloor x \rfloor} \rfloor$, and so on down the line. For instance, the continued fraction for π begins

$$3+\cfrac{1}{7+\cfrac{1}{15+\cfrac{1}{1+\cfrac{1}{292+\cdots}}}}$$

The kth continued fraction approximation or the kth convergent of x, which we denote by $\frac{p_k}{q_k}$, is what we get when we truncate x's continued fraction after the kth plus sign. For example, the convergents to π are 3, $\frac{22}{7}$, $\frac{333}{106}$, $\frac{355}{113}$, and so on. These approximations have many interesting properties, the first of which is the recurrence for p_n and q_n we proved in Week 8. We have

$$p_n = a_n p_{n-1} + p_{n-2}$$

and

$$q_n = a_n q_{n-1} + q_{n-2}$$

As a direct result of these recurrences (plus the fact that all the a_ns, q_ns, and p_ns are positive integers), we can say $q_{k+1} > q_k$ and $p_{k+1} > p_k$ for all k. It is often said that continued fractions provide the "best" approximations to a number, but the sense in which this is true is somewhat subtle. The property of continued fractions that justifies calling them the "best" approximations is the following:

Theorem 4.1. *let $\frac{p_n}{q_n}$ be the nth convergent of x. Then for all p, q such that*

$$|qx-p| < |q_n x - p_n|$$

we must have $q \geq q_{n+1}$. Equivalently, if $q < q_{n+1}$, then

$$|qx-p| \geq |q_n x - p_n|$$

Unfortunately the proof is a little too long and a little too unrelated to the Flint Hills Series to include here, but a version of it can be found at [3].

Corollary 4.1.1. *For all p, q with $q \leq q_n$, we have $|x - \frac{p_n}{q_n}| \leq |x - \frac{p}{q}|$.*

Proof. Suppose we have some p, q with $q \leq q_n$. Then we must have $q < q_{n+1}$, which by Theorem 4.1 implies

$$|qx-p| \geq |q_n x - p_n|$$

Using $q \leq q_n$ we see that $\frac{1}{q} \geq \frac{1}{q_n}$. Therefore

$$\frac{1}{q}|qx-p| \geq \frac{1}{q_n}|q_n x - p_n|$$

$$|x-\frac{p}{q}| \geq |x-\frac{p_n}{q_n}|$$

□

So, the kth convergent of x is the closest rational number to x out of all rationals with a denominator less than or equal to q_k. Out of the rationals with denominators between q_k and q_{k+1}, $\frac{p_k}{q_k}$ is not necessarily the closest to x. (For instance, $\frac{13}{4}$, with denominator between $q_1 = 1$ and $q_2 = 7$, is closer to π than $\frac{p_1}{q_1} = \frac{3}{1}$.) However, Theorem 4.1 tells us that the inequality $|qx - p| \geq |q_k x - p_k|$ holds for $q_k \leq q < q_{k+1}$, so we can still consider $\frac{p_k}{q_k}$ the "closest" approximation to x in that sense. There is one more property of continued fractions that we will be using:

Theorem 4.2. *Let the kth convergent of x be $\frac{p_k}{q_k}$. Then*

$$|q_k x - p_k| > \frac{1}{2q_{k+1}}$$

Proof. Since $p_k \neq p_{k+1}$ and $q_k \neq q_{k+1}$, we know that $|q_{k+1}p_k - p_{k+1}q_k|$ is a positive integer. So,

$$\frac{1}{q_k q_{k+1}} \leq \frac{|q_{k+1}p_k - p_{k+1}q_k|}{q_k q_{k+1}} = \left|\frac{p_k}{q_k} - \frac{p_{k+1}}{q_{k+1}}\right| = \left|\frac{p_k}{q_k} - x + x - \frac{p_{k+1}}{q_{k+1}}\right|$$

By the triangle inequality,

$$\left|\frac{p_k}{q_k} - x + x - \frac{p_{k+1}}{q_{k+1}}\right| \leq \left|\frac{p_k}{q_k} - x\right| + \left|x - \frac{p_{k+1}}{q_{k+1}}\right|$$

Now suppose for the sake of contradiction that $|q_k x - p_k| \leq \frac{1}{2q_{k+1}}$. Then we have

$$\left|\frac{p_k}{q_k} - x\right| \leq \frac{1}{2q_k q_{k+1}}$$

and

$$\left|\frac{p_{k+1}}{q_{k+1}} - x\right| \leq \frac{1}{2q_{k+1}q_{k+2}} \leq \frac{1}{2q_{k+1}^2}$$

So,

$$\frac{1}{q_k q_{k+1}} \leq \frac{1}{2q_k q_{k+1}} + \frac{1}{2q_{k+1}^2}$$

$$\frac{1}{2q_k q_{k+1}} \leq \frac{1}{2q_{k+1}^2}$$

$$\frac{1}{q_k} \leq \frac{1}{q_{k+1}}$$

$$q_k \geq q_{k+1}$$

and we know that's not true. Therefore, $|q_k x - p_k| > \frac{1}{2q_{k+1}}$. □

5 The $u$liπnt $v$ills $\sum$eries

In another paper due to Chakhkiev, Ziroyan, Tretyakov, and Mouhammad [2], we find some interesting things to say about an even more general series

$$\sum_{n=1}^{\infty} \frac{1}{n^u |\sin^v(\pi n x)|}$$

Whether this converges is again related to the irrationality measure of π. If x is rational, then this doesn't converge: we'll eventually have $nx \in \mathbb{Z}$, giving us a denominator of 0. So let's have x be irrational. By Lemma 3.1, we have

$$\sin(\pi n x) = \sin(\pi n x - m\pi) > \frac{2}{\pi} |\pi n x - m\pi|$$

$$\sin(\pi n x) > 2 |nx - m|$$

Denote by $\frac{p_k}{q_k}$ the kth continued fraction approximation of x. By Theorem 4.1, for all m and all $n < q_k$, we have

$$|nx - m| \geq |q_k x - p_k|$$

Therefore, we can replace the numbers in the denominator of our series with smaller numbers like so:

$$\sum_{n=1}^{\infty} \frac{1}{n^u |\sin^v(\pi n x)|} \leq \sum_{n=1}^{\infty} \frac{1}{n^u 2^v |nx - m|^v} \leq \sum_{k=1}^{\infty} \sum_{n=q_k}^{q_{k+1}} \frac{1}{q_k^u 2^v |q_{k+1}x - p_{k+1}|^v}$$

By Theorem 4.2,

$$\sum_{k=1}^{\infty} \sum_{n=q_k}^{q_{k+1}} \frac{1}{q_k^u 2^v |q_{k+1}x - p_{k+1}|^v} \leq \sum_{k=1}^{\infty} \sum_{n=q_k}^{q_{k+1}} \frac{1}{q_k^u 2^v |\frac{1}{2} q_{k+2}^{-1}|^v} \leq \sum_{k=1}^{\infty} \sum_{n=q_k}^{q_{k+1}} \frac{q_{k+2}^v}{q_k^u}$$

Now, since we're not using n anymore and $q_{k+1} - q_k < q_{k+1}$, we get rid of the inner summation:

$$\sum_{k=1}^{\infty} \sum_{n=q_k}^{q_{k+1}} \frac{{q_{k+1}}^v}{q_k^u} \leq \sum_{k=1}^{\infty} \frac{q_{k+1} {q_{k+1}}^v}{q_k^u}$$

Therefore,

$$\sum_{n=1}^{\infty} \frac{1}{n^u |\sin^v(\pi n x)|} \leq \sum_{k=1}^{\infty} \frac{q_{k+2}^{v+1}}{q_k^u}$$

We can reduce this even further using some facts about continued fractions (specifically, $q_n > q_{n-1}$ and $q_{n+1} = a_n q_n + q_{n-1}$):

$$q_{k+2} = a_{k+1} q_{k+1} + q_k = a_{k+1} a_k q_k + a_{k+1} q_{k-1} + q_k \leq (a_{k+1} a_k + a_{k+1} + 1) q_k \leq 2 a_{k+1} a_k q_k$$

So,

$$\sum_{k=1}^{\infty} \frac{q_{k+2}^{v+1}}{q_k^u} \leq \sum_{k=1}^{\infty} \frac{2(a_k a_{k+1})^{v+1} q_k^{v+1}}{q_k^u} = \sum_{k=1}^{\infty} \frac{2(a_k a_{k+1})^{v+1}}{q_k^{u-v-1}}$$

Now we can say some things about the series converging. Intuitively, we expect the denominator to grow much faster than the numerator when $u - v - 1 > 0$. Indeed, though the proof is slightly beyond the scope of this paper, the authors found that if $u > v + 1$ then the series converges "almost everywhere" in a measure-theoretic sense, meaning everywhere except in a set of measure zero. In other words, there are very few x such that the series

$$\sum_{n=1}^{\infty} \frac{1}{n^u |\sin^v(\pi n x)|}$$

diverges.

References

[1] Max A. Alekseyev. On convergence of the flint hills series. http://arxiv.org/abs/1104.5100/ (2011/04/27).

[2] Magomed A. Chakhkiev, Manya A. Ziroyan, N. P. Tretyakov, and Saif A. Mouhammad. On the convergence of some special series. *International Journal of Mathematical Analysis*, 10(12):573–577, 2016.

[3] Bruce Ikenaga. Approximation by rational numbers. http://sites.millersville.edu/bikenaga/number-theory/approximation-by-rationals/approximation-by-rationals.html.

[4] Doron Zeilberger and Wadim Zudilin. The irrationality measure of π is at most 7.103205334137... https://sites.math.rutgers.edu/ zeilberg/mamarim/mamarimPDF/pimeas.pdf (2020/01/07).