EXPONENTIAL GENERATING FUNCTIONS

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1. ABSTRACT

In this paper we'll discuss ordinary generating functions and exponential generating functions. First we'll show some examples of each and techniques to find explicit functions. Then, we'll incorporate calculus into our discussion of generating functions by discussing the significance of integrating and differentiating generating functions. We'll then go on to discuss the multiplication rule of different exponential and ordinary generating functions and explore connections with the binomial theorem. Finally we'll show why these functions are important, showing their application to mathematics through the tangent numbers, the Bernoulli numbers, the Euler numbers and the Genocchi numbers and interesting combinatorial interpretations of each of these numbers.

2. Ordinary Generating Functions

Let's introduce the concept of an ordinary generating function.

Definition 2.1. Given a sequence of numbers (a_n), the *generating function* for that sequence is the function given by the summation

$$f(x) = \sum_{n=0}^{\infty} a_n x^n$$

Let's look at some examples of generating functions now.

Example. The sequence $(a_n) = (1, \ldots, 1)$ has the generating function

$$f(x) = \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} x^n$$

From the definition of the of a geometric series we get that

$$f(x) = \frac{1}{1-x}$$

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Example. Similarly, let $(a_n) = (1, 2, 3 \ldots, n-1, n)$ which gives us the equation

$$f(x) = \sum_{n=0}^{\infty} n x^n$$

Now, this sequence seems a bit difficult because it's not exactly a geometric series. Let's try re-writing this in a form that's more recognizable. Factoring out an x we get that $f(x) = x\sum_{n=0}^{\infty} nx^{n-1}$. Now, nx^{n-1} just looks like the derivative of $g(x) = x^n$ so this

Date: March 2020.

means we get our generating function to be $f(x) = x\sum_{n=0}^{\infty} nx^{n-1}$ and then we get that $\int \frac{f(x)}{x}\,dx = \int \sum_{n=0}^{\infty} nx^{n-1}\,dx$ and swapping the integral and summation we get

$$\int \frac{f(x)}{x}\,dx = \sum_{n=0}^{\infty} \int nx^{n-1}\,dx$$

This then becomes

$$\int \frac{f(x)}{x}\,dx = \sum_{n=0}^{\infty} x^n$$

From our last example we see that summation becomes $\int \frac{f(x)}{x}\,dx = \frac{1}{1-x}$. Then, differentiating each side we get that

$$\frac{1}{x}f(x) = \frac{d}{dx}\left(\frac{1}{1-x}\right) = \frac{-1}{(1-x)^2}$$

$$f(x) = \frac{-x}{(1-x)^2}$$

Example. Another example of a generating function is the generating function for the recurrence relation of the Fibonacci sequence where

$$F_0 = 0, F_1 = 1, F_2 = 1, F_n = F_{n-1} + F_{n-2}$$

$$f(x) = \sum_{n=1}^{\infty} F_n x^n$$

We know that

$$F(x) = x + \sum_{n=2}^{\infty} F_n x^n$$

$$F(x) = x + \sum_{n=2}^{\infty} (F_{n-1} + F_{n-2})x^n$$

$$F(x) = x + \sum_{n=2}^{\infty} F_{n-1}x^n + \sum_{n=2}^{\infty} F_{n-2}x^n$$

$$\sum_{n=2}^{\infty} F_{n-1}x^n = x\sum_{n=2}^{\infty} F_{n-1}x^{n-1} = x\sum_{n=1}^{\infty} F_n x^n = xF(x)$$

$$\sum_{n=2}^{\infty} F_{n-2}x^n = x^2\sum_{n=2}^{\infty} F_{n-2}x^{n-2} = x^2\sum_{n=0}^{\infty} F_n x^n = x^2F(x)$$

$$F(x) = x + xF(x) + x^2F(x)$$

$$F(x) - xF(x) - x^2F(x) = x$$

$$F(x)(1 - x - x^2) = x$$

Hence

$$F(x) = \frac{x}{1 - x - x^2}$$

3. Exponential Generating Functions

Now that we've seen several examples of ordinary generating functions and how to find them given their sequences. Now, we're going to go on to study exponential generating functions.

Definition 3.1. An *exponential generating function* of a sequence (a_n) is a function $f(x)$ of the form

$$f(x) = \sum_{n=0}^{\infty} \frac{a_n}{n!} x^n$$

Example. The exponential generating function of the sequence $(a_n) = (1, \ldots, 1)$ is the function

$$f(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x$$

We call this function an exponential generating function because with the factorial it generates exponential functions.

Example.

$$a_n = 2^n$$

$$f(x) = \sum_{n=0}^{\infty} \frac{2^n}{n!} x^n$$

$$f(x) = \sum_{n=0}^{\infty} \frac{(2x)^n}{n!}$$

$$f(x) = e^{2x}$$

Example.

$$a_n = p_n$$

Where p_n is the number of permutations of a set n.
From probability we can deduce that $|p_n| = n!$ This means

$$f(x) = \sum_{n=0}^{\infty} \frac{a_n}{n!} x^n$$

$$f(x) = \sum_{n=0}^{\infty} \frac{p_n}{n!} x^n$$

$$f(x) = \sum_{n=0}^{\infty} \frac{n!}{n!} x^n$$

$$f(x) = \sum_{n=0}^{\infty} x^n$$

$$f(x) = \frac{1}{1-x}$$

This essentially means that the exponential generating function for the number of permutations of a set of a size n is the same as the ordinary generating function for the sequence of $\{1, 1, \ldots\}$.

Now, let's generate e^x.

Proof.

$$a_n = 1 \ \forall n$$

This gives us the official exponential generating function of

$$f(x) = \sum_{n=0}^{\infty} \frac{a_n}{n!} x^n$$

$$f(x) = \sum_{n=0}^{\infty} \frac{1}{n!} x^n$$

Ignoring our knowledge of Taylor Series we can prove that this is the exponential generating function of e^x because

$$f'(x) = \sum_{n=0}^{\infty} \frac{1}{n!} x^n$$

This means that $f(x) = f'(x)$ which gives us the differential equation $\frac{df}{dx} = f$ and then

$$\int \frac{1}{f}\, df = \int 1\, dx$$

This gives us

$$\ln f(x) = x \to f(x) = e^x$$

∎

Let's now prove the generating functions for $\sinh x$ and $\cosh x$

Example. Let $a_n = \frac{1+(-1)^n}{2}$

$$f(x) = \sum_{n=0}^{\infty} \frac{a_n}{n!} x^n$$

$$f(x) = \sum_{n=0}^{\infty} \frac{1+(-1)^n}{2 \cdot (n!)} x^n$$

$$f(x) = \sum_{n=0}^{\infty} \frac{1}{2} (\frac{x^n}{n!} + \frac{(-1)^n (x)^n}{n!})$$

$$f(x) = \frac{1}{2} \sum_{n=0}^{\infty} (\frac{x^n}{n!} + \frac{(-x)^n}{n!})$$

$$f(x) = \frac{1}{2} \sum_{n=0}^{\infty} (\frac{1}{n!} + \frac{(-1)^n x^n}{n!})$$

$$f(x) = \frac{1}{2} \sum_{n=0}^{\infty} \frac{1}{n!} x^n + \frac{1}{2} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} x^n$$

$$f(x) = \frac{e^x}{2} + \frac{e^{-x}}{2}$$

$$f(x) = \frac{e^x + e^{-x}}{2}$$
$$f(x) = \cosh(x)$$

Moving on, we can find the generating function for $\sinh(x)$

Example.

$$\sinh(x) = \frac{e^x - e^{-x}}{2}$$
$$\sinh(x) = \frac{1}{2}(e^x - e^{-x})$$
$$\sinh(x) = \frac{1}{2}(\sum_{n=0}^{\infty} \frac{1}{n!}x^n - \sum_{n=0}^{\infty} \frac{(-1)^n}{n!}x^n)$$
$$\sinh(x) = \frac{1}{2}(\sum_{n=0}^{\infty} \frac{1}{n!}x^n - \frac{(-1)^n}{n!}x^n)$$
$$\sinh(x) = \frac{1}{2}\sum_{n=0}^{\infty}(\frac{1}{n!} - \frac{(-1)^n}{n!})x^n$$
$$\sinh(x) = \sum_{n=0}^{\infty}(\frac{1-(-1)^n}{2 \cdot n!})x^n$$

Hence, the generating function for $\sinh(x)$ is $a_n = \frac{1-(-1)^n}{2}$

4. Operations on Exponential Generating Functions

Definition 4.1. *Derivative of an Exponential Generating Function* Let $f(x)$ be the exponential generating function for the sequence (a_n)

$$f(x) = \sum_{n=0}^{\infty} \frac{a_n}{n!}x^n$$
$$\frac{d}{dx}(f(x)) = \frac{d}{dx}\sum_{n=0}^{\infty} \frac{a_n}{n!}x^n$$
$$f'(x) = \sum_{n=1}^{\infty} \frac{a_n}{(n-1)!}x^{n-1}$$

This is also equivalent to the exponential generating function of $(a_n)_{n\neq 0}$ This is equivalent to just shifting over the sequence to the left.

Let's look at an example of this:

Example.

$$a_n = \left(\frac{1}{3}\right)^n$$
$$f(x) = \sum_{n=0}^{\infty} \frac{1}{3^n n!}x^n$$

$$f(x) = \sum_{n=0}^{\infty} \frac{1}{n!}\left(\frac{x}{3}\right)^n$$
$$f'(x) = \sum_{n=1}^{\infty} \frac{a_n}{(n-1)!}x^{n-1}$$
$$f'(x) = \sum_{n=1}^{\infty} \frac{1}{(n-1)!}\frac{x^{n-1}}{3^n}$$
$$f'(x) = \frac{1}{3}e^{\frac{x}{3}}$$
$$f(x) = e^{\frac{x}{3}}$$
$$f'(x) = \frac{1}{3}e^{\frac{x}{3}}$$

Definition 4.2. *Integral of an Exponential Generating Function* Let $(a_n)_{n\geq 0}$ be a sequence of numbers and $f(x) = \sum_{n=0}^{\infty} a_n x^n$

$$\int_0^x f(t)dt = \int_0^x \sum_{n=0}^{\infty} \frac{a_n}{n!}x^n dx$$
$$\int_0^x f(t)dt = \sum_{n=0}^{\infty} \int \frac{a_n}{n!}x^n dx$$
$$\int_0^x f(t)dt = \sum_{n=0}^{\infty} a_n \int x^n dx$$
$$\int_0^x f(t)dt = \sum_{n=0}^{\infty} \frac{a_n}{n!}\frac{x^{n+1}}{n+1}$$
$$\int_0^x f(t)dt = \sum_{n=0}^{\infty} \frac{a_n}{(n+1)!}x^{n+1}$$

This is equivalent to shifting the sequence to the right.

Let's try an example with an exponential generating function.

Example.

$$f(x) = \sum_{n=1}^{\infty} \frac{3^n+1}{n!}x^n$$
$$\int f(x)dx = \int \sum_{n=1}^{\infty} \frac{3^n+1}{n!}x^n dx$$
$$\int f(x)dx = \sum_{n=1}^{\infty} \frac{3^n+1}{(n+1)!}x^{n+1}$$
$$f(x) = e^{3x} + e^x$$
$$\int f(x)dx = \frac{e^{3x}}{3} + e^x$$

Definition 4.3. *Multiplication Rule* Let $A(x)$ and $B(x)$ be two exponential generating functions such that

$$A(x) = \sum_{n=0}^{\infty} \frac{a_n}{n!} x^n$$

$$B(x) = \sum_{n=0}^{\infty} \frac{b_n}{n!} x^n$$

Now, we define the product of these two functions to be

$$C(x) = A(x) \cdot B(x)$$

$$C(x) = \sum_{n=0}^{\infty} \frac{c_n}{n!} x^n$$

$$c_n = \sum_{k=0}^{n} \binom{n}{k} a_k \cdot b_{n-k}$$

$$C(x) = A(x)B(x) = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \binom{n}{k} a_k \cdot b_{n-k} \frac{x^n}{n!}$$

$$C(x) = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{a_k \cdot b_{n-k}}{(n-k)! \cdot k!} x^n$$

Looking at this we see that multiplying two generating function is very similar to the binomial theorem which can help us understand the mechanics of what is going on when multiplying power series.

5. Important Exponential Generating Functions

Now, we'll move on to examples of exponential generating functions that are applicable to other fields and different areas of mathematics.

5.1. Tangent Numbers.

First, we'll talk about tangent numbers.

Definition 5.1. *Tangent Numbers* Tangent Numbers are numbers given by the form

$$T_n = \frac{2^{2n}(2^{2n}-1)|B_{2n}|}{2n}$$

Here, B_{2n} denotes an even Bernoulli number where a Bernoulli number is defined as follows:

Definition 5.2. *Bernoulli Number* The Bernoulli Numbers denoted (B_n) are the coefficients of the exponential generating function for $\frac{x}{e^x-1}$. In other words,

$$\frac{x}{e^x - 1} = \sum_{n=0}^{\infty} \frac{B_n x^n}{n!}$$

Another way to write the Bernoulli numbers is so that they can be written in terms of the contour integral

$$B_n = \frac{n!}{2\pi i} \oint_C \frac{z}{e^z - 1} \frac{dz}{z^{n+1}}$$

where the given contour is a circle enclosing the origin, with radius less than 2π, and is traversed in a counter-clockwise direction.

Moving on, we can find the exponential generating function for the tangent function, $\tan(x)$ which yields us the following.

$$\tan(x) = \sum_{n=0}^{\infty} \frac{(-1)^{n-1}2^{2n}(2^{2n}-1)B_{2n}}{(2n)!}x^{2n-1}$$

This then further simplifies to

$$\tan(x) = \sum_{n=1}^{\infty} \frac{T_n}{(2n-1)!}x^{2n-1}$$

And pulling out numbers we see that the exponential generating function for $\tan(x)$ can be written as the following polynomial.

$$\tan(x) = x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \frac{17}{315}x^7 + \cdots$$

This then gives us the series for the Tangent Numbers. Now, let's move on and discuss another similar sequence of numbers.

5.2. **Euler Numbers.** Similar to Bernoulli numbers and Tangent numbers, Euler numbers can also we written as the coefficients for a generating function.

Here, we get that an Euler number is the coefficient of an exponential generating function for the function $\frac{1}{\cosh(x)}$ or $\text{sech}(x)$.

Definition 5.3. *Euler Numbers* The Euler Numbers, denoted by E_n have the form

$$\text{sech}(x) = \sum_{n=0}^{\infty} \frac{E_n}{n!}x^n$$

Following from here gives us the formula that

$$E_n = 2^n E_n(\frac{1}{2})$$

and here, $E_n(x)$ are the Euler Polynomials. These polynomials are given by the generating function

$$\frac{2e^{xt}}{e^t+1} = \sum_{n=0}^{\infty} E_n(x)\frac{t^n}{n!}$$

Moving on from Euler Polynomials we can see other properties of Euler numbers which are extremely useful and interesting.

We can write even Euler Numbers in two very efficient asymptotic sequences.

(1) $E_{2n} \approx (-1)^n 8\sqrt{\frac{n}{\pi}}(\frac{4n}{\pi e})^{2n}$
(2) $E_{2n} \approx (-1)^n 8\sqrt{\frac{n}{\pi}}(\frac{4n}{\pi e} \cdot \frac{480n^2+9}{480n^2-1})^{2n}$

Furthermore, if we can use another interesting theorem to expand a finite product of Euler Numbers.

Theorem 5.4. *$(E-i)^n = 0$ if n is even , $(E-i)^n = -iT_{\frac{n+1}{2}}$ if n is odd and here the term E^k is interpreted as $|E_k|$ and*

Let's look at another couple interesting theorems.

Theorem 5.5. *If E_k and E_j are two elements of the sequence E_n then*

$$E_k \equiv E_j \mod 2^n$$

if and only if

$$k \equiv j \mod 2^n$$

And our final theorem for Euler Numbers is courtesy of Shanks.

Theorem 5.6.

$$E_{2n} = 2 \cdot (-1)^n \cdot (2n)! L_1(2n+1) \cdot (\frac{2}{\pi})^{2n+1}$$

Here, L represents an L function.

Finally, we'll now begin to talk about our final number sequence related to generating functions, Genocchi Numbers.

5.3. **Genocchi Numbers.**

Definition 5.7. *Genocchi Number* Genocchi numbers are the numbers in the sequence G_n that satisfy the generating function

$$\frac{2x}{e^x + 1} = \sum_{n=1}^{\infty} \frac{G_n}{n!} x^n$$

We can look at some interesting properties of Genocchi numbers related to Bernoulli numbers and Euler polynomials which we talked about earlier. This brings all of the interesting number sequences together at once.

Theorem 5.8.

$$G_{2n} = 2(1 - 2^{2n}) B_{2n}$$
$$G_{2n} = 2n E_{2n-1}(0)$$

6. Bibliography

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