COMPLEX ANAYLSIS

JERRY SUN AND WILLIAM REN

1. Introduction

We will talk about some of the very basic definitions of Complex analysis such as, analytic functions, meromorphic functions, singularities,and homomorphic functions. We will talk about the Residue Theorem and the Cauchy Integral formula and do some example problems. We will also talk about Laurent Series and how they are like Taylor Series except they include negative terms as well. Then we will start showing how we can use complex analysis techniques to evaluate infinite series and infinite products. We will then talk about entire functions and product representations of functions.

Entire function:In complex analysis, an entire function, also called an integral function, is a complex-valued function that is holomorphic at all finite points over the whole complex plane

One of the central tools in complex analysis is the line integral. The line integral around a closed path of a function that is holomorphic everywhere inside the area bounded by the closed path is always zero, as is stated by the Cauchy integral theorem. The values of such a holomorphic function inside a disk can be computed by a path integral on the disk's boundary (as shown in Cauchy's integral formula). Path integrals in the complex plane are often used to determine complicated real integrals, and here the theory of residues among others is applicable (see methods of contour integration). A "pole" (or isolated singularity) of a function is a point where the function's value becomes unbounded, or "blows up". If a function has such a pole, then one can compute the function's residue there, which can be used to compute path integrals involving the function; this is the content of the powerful residue theorem. The remarkable behavior of holomorphic functions near essential singularities is described by Picard's Theorem. Functions that have only poles but no essential singularities are called meromorphic. Laurent series are the complex-valued equivalent to Taylor series, but can be used to study the behavior of functions near singularities through infinite sums of more well understood functions, such as polynomials.

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A bounded function that is holomorphic in the entire complex plane must be constant; this is Liouville's theorem. It can be used to provide a natural and short proof for the fundamental theorem of algebra which states that the field of complex numbers is algebraically closed.
If a function is holomorphic throughout a connected domain then its values are fully determined by its values on any smaller subdomain. The function on the larger domain is said to be analytically continued from its values on the smaller domain. This allows the extension of the definition of functions, such as the Riemann zeta function, which are initially defined in terms of infinite sums that converge only on limited domains to almost the entire complex plane. Sometimes, as in the case of the natural logarithm, it is impossible to analytically continue a holomorphic function to a non-simply connected domain in the complex plane but it is possible to extend it to a holomorphic function on a closely related surface known as a Riemann surface.
All this refers to complex analysis in one variable. There is also a very rich theory of complex analysis in more than one complex dimension in which the analytic properties such as power series expansion carry over whereas most of the geometric properties of holomorphic functions in one complex dimension (such as conformality) do not carry over. The Riemann mapping theorem about the conformal relationship of certain domains in the complex plane, which may be the most important result in the one-dimensional theory, fails dramatically in higher dimensions. A major use of certain complex spaces is in quantum mechanics as wave functions.

Complex functions that are differentiable at every point of an open subset Ω of the complex plane are said to be holomorphic on Ω. In the context of complex analysis, the derivative of f at z0 is defined to be
$f'(z_0) = \lim z \to z0 \frac{f(z)-f(z_0)}{z-z0}, z \in C$. $f'(z0) = \lim z \to z0 \frac{f(z)-f(z0)}{z-z0}, \quad z \in C$

2. Definitions

Definition 2.1. Holomorphic Functions:In mathematics, a holomorphic function is a complex-valued function of one or more complex variables that is, at every point of its domain, complex differentiable in a neighbourhood of the point. The existence of a complex derivative in a neighbourhood is a very strong condition, for it implies that any holomorphic function is actually infinitely differentiable and equal, locally, to its own Taylor series (analytic). Holomorphic functions are the central objects of study in complex analysis.

Though the term analytic function is often used interchangeably with "holomorphic function", the word "analytic" is defined in a broader sense to denote any function (real, complex, or of more general type) that can be written as a convergent power series in a neighbourhood of each point in its domain. The fact that all holomorphic functions are complex analytic functions, and vice versa, is a major theorem in complex analysis.[1]

Holomorphic functions are also sometimes referred to as regular functions.[2] A holomorphic function whose domain is the whole complex plane is called an entire function. The phrase "holomorphic at a point $z0$" means not just differentiable at z0, but differentiable everywhere within some neighbourhood of z0 in the complex plane. Given a complex-valued function f of a single complex variable, the derivative of f at a point z0 in its domain is defined by the limit[3]

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}.f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}.$$

This is the same as the definition of the derivative for real functions, except that all of the quantities are complex. In particular, the limit is taken as the complex number z approaches z0, and must have the same value for any sequence of complex values for z that approach z0 on the complex plane. If the limit exists, we say that f is complex-differentiable at the point z0. This concept of complex differentiability shares several properties with real differentiability: it is linear and obeys the product rule, quotient rule, and chain rule.[4]

If f is complex differentiable at every point $z0$ in an open set U, we say that f is holomorphic on U. We say that f is holomorphic at the point z0 if f is complex differentiable on some neighbourhood of z0.[5] We say that f is holomorphic on some non-open set A if it is holomorphic in an open set containing A. As a pathological non-example, the function given by f(z) = —z—2 is complex differentiable at exactly one point (z0 = 0), and for this reason, it is not holomorphic at 0 because there is no open set around 0 on which f is complex differentiable.

Definition 2.2. Memeromoprhic Functions In complex analysis, an entire function, also called an integral function, is a complex-valued function that is holomorphic at all finite points over the whole complex plane. Typical examples of entire functions are polynomials and the exponential function, and any finite sums, products and compositions of these, such as the trigonometric functions sine and cosine and their hyperbolic counterparts sinh and cosh, as well as derivatives and integrals of entire functions such as the error function. If an entire function f(z) has a root at w, then f(z)/(zw), taking the limit value at w, is an entire function. On the other hand, neither the natural logarithm nor the square root is an entire function, nor can they be continued analytically to an entire function.

A transcendental entire function is an entire function that is not a polynomial.

Definition 2.3. Poles and Zeros In complex analysis (a branch of mathematics), zeros of holomorphic functions—which are points z where f(z) = 0—play an important role.

For meromorphic functions, particularly, there is a duality between zeros and poles. A function f of a complex variable z is meromorphic in the neighbourhood of a point z0 if either f or its reciprocal function 1/f is holomorphic in some neighbourhood of z0 (that is, if f or 1/f is differentiable in a neighbourhood of z0). If z0 is a zero of 1/f , then it is a pole of f.

Thus a pole is a certain type of singularity of a function, nearby which the function behaves relatively regularly, in contrast to essential singularities, such as 0 for the logarithm function, and branch points, such as 0 for the complex square root function.

Definition 2.4. Laurent Series In mathematics, the Laurent series of a complex function f(z) is a representation of that function as a power series which includes terms of negative degree. It may be used to express complex functions in cases where a Taylor series expansion cannot be applied. The Laurent series was named after and first published by Pierre Alphonse Laurent in 1843. Karl Weierstrass may have discovered it first in a paper written in 1841, but it was not published until after his death.[1]

3. Theorems

Theorem 3.1. *Cauchy Goursat Theorem In mathematics, the Cauchy integral theorem (also known as the Cauchy–Goursat theorem) in complex analysis, named after Augustin-Louis*

Cauchy (and Édouard Goursat), is an important statement about line integrals for holomorphic functions in the complex plane. Essentially, it says that if two different paths connect the same two points, and a function is holomorphic everywhere in between the two paths, then the two path integrals of the function will be the same. The theorem is usually formulated for closed paths as follows: let U be an open subset of C which is simply connected, let f : U → C be a holomorphic function, and let γ be a rectifiable path in U whose start point is equal to its end point. Then

$$\oint_\gamma f(z)\,dz = 0. \oint_\gamma f(z)\,dz = 0.$$

Theorem 3.2. *Cauchy Integral Theorem Let U be an open subset of the complex plane C, and suppose the closed disk D defined as*

$$D = \left\{ z : |z - z_0| \leq r \right\}$$

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is completely contained in U. Let f : U → C be a holomorphic function, and let be the circle, oriented counterclockwise, forming the boundary of D. Then for every a in the interior of D,

$$f(a) = \frac{1}{2\pi i} \oint_\gamma \frac{f(z)}{z-a}\,dz. f(a) = \frac{1}{2\pi i} \oint_\gamma \frac{f(z)}{z-a}\,dz.$$

Theorem 3.3. *Residue Theorem Let U be a simply connected open subset of the complex plane containing a finite list of points a1, ..., an, and f a function defined and holomorphic on U{a1, ..., an. Let be a closed rectifiable curve in U which does not meet any of the ak, and denote the winding number of around a_k by $I(, a_k)$. The line integral of f around is equal to 2i times the sum of residues of f at the points, each counted as many times as winds around the point:*

$$\oint_\gamma f(z)\,dz = 2\pi i \sum_{k=1}^{n} \mathrm{I}(\gamma, a_k)\,\mathrm{Res}(f, a_k). \oint_\gamma f(z)\,dz = 2\pi i \sum_{k=1}^{n} \mathrm{I}(\gamma, a_k)\,\mathrm{Res}(f, a_k).$$

If is a positively oriented simple closed curve, I(, ak) = 1 if ak is in the interior of , and 0 if not, so

$$\oint_\gamma f(z)\,dz = 2\pi i \sum \mathrm{Res}(f, a_k) \oint_\gamma f(z)\,dz = 2\pi i \sum \mathrm{Res}(f, a_k)$$

with the sum over those a_k inside .

4. Application to Infinite Sums

The integral

$$\int_{-\infty}^{\infty} \frac{e^{itx}}{x^2+1}\,dx$$

The contour C. arises in probability theory when calculating the characteristic function of the Cauchy distribution. It resists the techniques of elementary calculus but can be evaluated by expressing it as a limit of contour integrals.

Suppose t ¿ 0 and define the contour C that goes along the real line from a to a and then counterclockwise along a semicircle centered at 0 from a to a. Take a to be greater than 1, so that the imaginary unit i is enclosed within the curve. Now consider the contour integral

$$\int_C f(z)\,dz = \int_C \frac{e^{itz}}{z^2+1}\,dz. \int_C f(z)\,dz = \int_C \frac{e^{itz}}{z^2+1}\,dz.$$

Since eitz is an entire function (having no singularities at any point in the complex plane), this function has singularities only where the denominator z2 + 1 is zero. Since z2 + 1 = (z + i)(z i), that happens only where z = i or z = i. Only one of those points is in the region bounded by this contour. Because f(z) is

$$\frac{e^{itz}}{z^2+1} = \frac{e^{itz}}{2i}\left(\frac{1}{z-i} - \frac{1}{z+i}\right) \frac{e^{itz}}{z^2+1} = \frac{e^{itz}}{2i}\left(\frac{1}{z-i} - \frac{1}{z+i}\right)$$
$$= \frac{e^{itz}}{2i(z-i)} - \frac{e^{itz}}{2i(z+i)}, \qquad = \frac{e^{itz}}{2i(z-i)} - \frac{e^{itz}}{2i(z+i)},$$

the residue of f(z) at z = i is

$$\operatorname{Res}_{z=i} f(z) = \frac{e^{-t}}{2i}. \operatorname{Res}_{z=i} f(z) = \frac{e^{-t}}{2i}.$$

According to the residue theorem, then, we have

$$\int_C f(z)\,dz = 2\pi i \cdot \operatorname{Res}_{z=i} f(z) = 2\pi i \frac{e^{-t}}{2i} = \pi e^{-t}. \int_C f(z)\,dz = 2\pi i \cdot \operatorname{Res}_{z=i} f(z) = 2\pi i \frac{e^{-t}}{2i} = \pi e^{-t}.$$

The contour C may be split into a straight part and a curved arc, so that

$$\int_{\text{straight}} f(z)\,dz + \int_{\text{arc}} f(z)\,dz = \pi e^{-t} \int_{\text{straight}}$$

f(z) dz+$\int_{\text{arc}} f(z)\,dz = \pi e^{-t}\, andthus$

$$\int_{-a}^{a} f(z)\,dz = \pi e^{-t} - \int_{\text{arc}} f(z)\,dz. \int_{-a}^{a} f(z)\,dz = \pi e^{-t} - \int_{\text{arc}} f(z)\,dz.$$

Using some estimations, we have

$$\left|\int_{\text{arc}} \frac{e^{itz}}{z^2+1}\,dz\right| \le \pi a \cdot \sup_{\text{arc}} \left|\frac{e^{itz}}{z^2+1}\right| \le \pi a \cdot \sup_{\text{arc}} \frac{1}{|z^2+1|} \le \frac{\pi a}{a^2-1}, \left|\int_{\text{arc}} \frac{e^{itz}}{z^2+1}\,dz\right|$$
$$\le \pi a \cdot \sup_{\text{arc}} \left|\frac{e^{itz}}{z^2+1}\right| \le \pi a \cdot \sup_{\text{arc}} \frac{1}{|z^2+1|} \le \frac{\pi a}{a^2-1},$$

and

$$\lim_{a\to\infty} \frac{\pi a}{a^2-1} = 0. \lim_{a\to\infty} \frac{\pi a}{a^2-1} = 0.$$

The estimate on the numerator follows since t ¿ 0, and for complex numbers z along the arc (which lies in the upper halfplane), the argument of z lies between 0 and . So,

$$\left|e^{itz}\right| = \left|e^{it|z|(\cos\phi + i\sin\phi)}\right| = \left|e^{-t|z|\sin\phi + it|z|\cos\phi}\right| =$$

$$e^{-t|z|\sin\phi} \leq 1. \left|e^{itz}\right| = \left|e^{-t|z|\sin\phi + it|z|\cos\phi}\right| = e^{-t|z|\sin\phi} \leq 1.$$

Therefore,

$$\int_{-\infty}^{\infty} \frac{e^{itz}}{z^2+1}\, dz = \pi e^{-t}..$$

If t ¡ 0 then a similar argument with an arc C that winds around i rather than i shows that
The contour C.

$$\int_{-\infty}^{\infty} \frac{e^{itz}}{z^2+1}\, dz = \pi e^{t},$$

and finally we have

$$\int_{-\infty}^{\infty} \frac{e^{itz}}{z^2+1}\, dz = \pi e^{-|t|}.$$

5. References

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