FOURIER ANALYTIC METHODS FOR EVALUATING SERIES

ETHAN YANG

ABSTRACT. We give examples of series being evaluated using methods relating to Fourier series, with a special focus on evaluating even values of ζ using various methods.

1. Introduction

Fourier series provide an extremely powerful method for evaluating many infinite series. In this article, we start by evaluating series by plugging in values into Fourier series of various functions in section 2. In section 3, we prove Parseval's Theorem and use it to evaluate series such as

$$\sum_{n=1}^{\infty} \frac{\sin^2(nd)}{n^2}.$$

Finally, in section 4, we describe the Poisson Summation Formula and use it to sum series of the form

$$\sum_{n=1}^{\infty} \frac{1}{y^2+n^2}.$$

Recall that the Fourier series representation of a periodic function $f(x)$ is given by the form:

$$f(x) = \sum_{m=0}^{\infty} a_m \cos mx + \sum_{n=1}^{\infty} b_n \sin nx,$$

We determine the coefficients with the following formulas.

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos kx\, dx, \qquad a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\, dx.$$

To compute the b coefficients, we have the following similar formula.

$$b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin kx\, dx.$$

2. Basic Examples

We can evaluate a lot of infinite series by just plugging numbers into the Fourier series of common functions. In class, we did this for the periodic functions $f(x) = x, x^2, x^3$ on the interval $(-\pi, \pi)$ as well as $f(x) = \frac{\pi - x}{2}$ on $(0, 2\pi)$. We give some more examples here. The first is a method on evaluating $\zeta(2m)$ for all even numbers. This relies on the Fourier expansion of the periodic version of $f_m(x) = x^{2m}$ on $(-\pi, \pi)$. The following method is referenced from [Rob99], with some errors corrected.

Theorem 2.1. *The Fourier series of $f_m(x) = x^{2m}, m \geq 1$ on $(-\pi, \pi)$, extended to be periodic, is:*

$$x^{2m} = c_{m,0} + \sum_{n=1}^{\infty} c_{m,n} \cos nx.$$

where

$$c_{m,0} = \frac{\pi^{2m}}{(2m+1)}, \text{ and } c_{m,n} = 2(-1)^n (2m)! \sum_{j=1}^{m} \frac{(-1)^{j-1}\pi^{2(m-j)}}{(2m+1-2j)!n^{2j}}$$

Proof. We directly compute the Fourier Series coefficients. We have that

$$b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} x^{2m} \sin kx = 0$$

since $x^{2m} \sin kx$ is an odd function. For the constant, we compute

$$c_{m,0} = a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} x^{2m} = \frac{\pi^{2m}}{(2m+1)}.$$

Finally, for the cosine coefficients, we use integration of parts twice.

$$\begin{aligned}
c_{m,n} = a_k &= \frac{1}{\pi}\int_{-\pi}^{\pi} x^{2m} \cos kx = \frac{2}{\pi}\int_{0}^{\pi} x^{2m} \cos kx \\
&= \frac{2}{\pi}\left[\left(x^{2m}\frac{\sin nx}{n}\right)_{x=0}^{x=\pi} - \frac{2m}{n}\int_0^{\pi} x^{2m-1} \sin nx\right] \\
&= \frac{-4m}{n\pi}\left[-\left(x^{2m-1}\frac{\cos nx}{n}\right)_{x=0}^{x=\pi} + \frac{2m-1}{n}\int_0^{\pi} x^{2m-2} \cos nx\right] \\
&= \frac{-4m}{n\pi}\left[(-1)^{n-1}\frac{\pi^{2m-1}}{n} + \frac{2m-1}{n}\frac{\pi}{2}c_{m-1,n}\right] \\
&= \frac{4m}{n^2}\left[(-1)^n \pi^{2m-2} - \frac{2m-1}{2}c_{m-1,n}\right].
\end{aligned}$$

The proof can be then completed with induction to derive the explicit formula for $c_{m,n}$. ∎

Now, we can derive a recurrence for even zeta values by plugging in $x = \pi$ into x^{2m} and its Fourier Series.

Theorem 2.2. *If $m \geq 1$, then $\zeta(2m) = a_m \pi^{2m}$, where*

$$\sum_{j=1}^{m} \frac{(-1)^{j-1}a_j}{(2m+1-2j)!} = \frac{m}{(2m+1)!}.$$

Proof. Plugging in $x = \pi$ and subtracting the constant on both sides, we have

$$\begin{aligned}
\pi^{2m} - \frac{\pi^{2m}}{(2m+1)} = \frac{2m\pi^{2m}}{(2m+1)} &= \sum_{n=1}^{\infty}(-1)^n c_{m,n} \\
&= \sum_{n=1}^{\infty} 2(-1)^{2n}(2m)! \sum_{j=1}^{m} \frac{(-1)^{j-1}\pi^{2(m-j)}}{(2m+1-2j)!n^{2j}} \\
&= 2(2m)! \sum_{j=1}^{m} \frac{(-1)^{j-1}\pi^{2(m-j)}}{(2m+1-2j)!}\zeta(2j).
\end{aligned}$$

Dividing by $2(2m)!$,

$$\sum_{j=1}^{m} \frac{(-1)^{j-1}\pi^{2(m-j)}}{(2m+1-2j)!}\zeta(2j) = \frac{m\pi^{2m}}{(2m+1)!}.$$

The exact format of the theorem can be proven with induction. ∎

Example. Computing $\zeta(6) = a_3\pi^6$, $\frac{a_1}{5!} - \frac{a_2}{3!} + \frac{a_3}{1!} = \frac{3}{7!}$, so $a_3 = -\frac{a_1}{5!} + \frac{a_2}{3!} + \frac{3}{7!} = \frac{1}{945}$, which is correct.

In general, plugging in values into Fourier Series of functions leads to being able to sum infinite series. For example, the Fourier series of $|\sin(\theta)|$ is

$$|\sin(\theta)| = \frac{2}{\pi} - \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{\cos 2n\theta}{4n^2-1}.$$

Plugging in $\theta = \frac{\pi}{2}$,

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{4n^2-1} = \frac{1}{2} - \frac{\pi}{4}.$$

3. Parseval's Theorem

We can evaluate more series by using Parseval's Theorem.

Theorem 3.1 (Parseval)**.** *Suppose that the Fourier series of f is*

$$f(x) = a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx).$$

Then

$$\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)^2\,dx = 2a_0^2 + \sum_{n=1}^{\infty}(a_n^2 + b_n^2).$$

Proof. We directly evaluate the integral on the left:

$$\begin{aligned}
\int_{-\pi}^{\pi} f(x)^2\,dx &= \int_{-\pi}^{\pi} a_0^2 + 2a_0\sum_{n=1}^{\infty}(a_n\cos nx + b_n \sin nx) + \left(\sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)\right)^2 dx \\
&= 2\pi a_0^2 + \int_{-\pi}^{\pi}\left(\sum_{n=1}^{\infty}(a_n\cos nx + b_n \sin nx)\right)^2 dx \\
&= 2\pi a_0^2 + \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\int_{-\pi}^{\pi} a_n a_m \cos nx \cos mx + b_n a_m \sin nx \cos mx \\
&\qquad + a_n b_m \cos nx \sin mx + b_n b_m \sin nx \sin mx\,dx \\
&= 2\pi a_0^2 + \sum_{n=1}^{\infty}(a_n^2\pi + b_n^2\pi).
\end{aligned}$$

The last integral was evaluated using the orthogonality relations

$$\int_{-\pi}^{\pi} \cos mx \cos nx \, dx = \begin{cases} 2\pi & m = n = 0, \\ \pi & m = n \neq 0, \\ 0 & \text{otherwise}, \end{cases}$$

$$\int_{-\pi}^{\pi} \sin mx \sin nx \, dx = \begin{cases} \pi & m = n \neq 0, \\ 0 & \text{otherwise}, \end{cases}$$

$$\int_{-\pi}^{\pi} \cos mx \sin nx \, dx = 0.$$

■

Like before, we can also use Parseval's Theorem to evaluate $\zeta(2m)$. In class, we saw how to find $\zeta(4)$ from $\zeta(2)$ by using Parseval's Theorem on the Fourier series of x^2. We can do something similar for $\zeta(6)$ by using Parseval's Theorem on x^3. The Fourier series of x^3 on $(-\pi, \pi)$ extended to be periodic is

$$x^3 = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} 2n^2\pi^2 + 12\,(-1)^n}{n^3} \sin nx.$$

Using Parseval's Theorem, we have

$$\frac{1}{\pi} \int_{-\pi}^{\pi} x^6 \, dx = \sum_{n=1}^{\infty} \left(\frac{2\,(-1)^{n+1} n^2\pi^2 + 12\,(-1)^n}{n^3} \right)^2.$$

Evaluating the integral and expanding the right side, we have

$$\begin{aligned} \frac{2\pi^6}{7} &= \sum_{n=1}^{\infty} \frac{4n^4\pi^4 - 48n^2\pi^2 + 144}{n^6} \\ &= 4\pi^4\zeta(2) - 48\pi^2\zeta(4) + 144\zeta(6). \end{aligned}$$

Solving for $\zeta(6)$, we get that $\zeta(6) = \frac{\pi^6}{945}$, which is correct. We can compute all even zeta values using this method.

We can also find sums of the form $\sum_{n=1}^{\infty} \frac{\sin^2 nd}{n^2}$ where $0 < d < \pi$, referenced from [Mat]. We use Parseval's Theorem on the following extended periodic function f and its Fourier Series

$$f(x) = \begin{cases} 1 \text{ if } 0 \le |x| \le d \\ 0 \text{ if } d \le |x| \le \pi \end{cases} = \frac{d}{\pi} + \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{\sin nd}{n} \cos nx.$$

Applying Parseval's Theorem, we have that

$$\frac{1}{2}\left(\frac{2d}{\pi}\right)^2 + \sum_{n=1}^{\infty} \frac{4\sin^2 nd}{n^2\pi^2} = \frac{2}{\pi} \int_0^{\pi} f^2(x) \, dx.$$

The right side evaluates to $\frac{2d}{\pi}$, and the left side evaluates to $\frac{2d^2}{\pi^2} + \frac{4}{\pi^2} \sum_{n=1}^{\infty} \frac{\sin^2 nd}{n^2}$. Solving for the sum, we have that

$$\sum_{n=1}^{\infty} \frac{\sin^2 nd}{n^2} = \frac{d(\pi - d)}{2}.$$

Notably, we can plug in $d = \frac{\pi}{2}$ to get that

$$\sum_{n=1}^{\infty} \frac{\sin^2 \frac{n\pi}{2}}{n^2} = \frac{\pi^2}{8}.$$

Since we know that

$$\sin^2 \frac{n\pi}{2} = \begin{cases} 0 \text{ if } 2 \mid n \\ 1 \text{ otherwise} \end{cases},$$

the series can also be written as

$$\sum_{k=0}^{\infty} \frac{1}{(2k+1)^2} = \frac{\pi^2}{8}.$$

4. Poisson Summation Formula

Finally, we'll show some interesting sums that can be computed using the Poisson summation formula, which is based on the Fourier transform of a function.

Definition 4.1. The *Fourier transform* of a function f, denoted $\hat{f}(x)$, is

$$\hat{f}(x) = \int_{-\infty}^{\infty} e^{-2\pi i x t} f(t)\, dt.$$

We'll ignore the exact hypotheses needed for the Poisson summation formula to be used for here.

Theorem 4.2 (Poisson Summation Formula)**.** *If f is a Schwartz function, then*

$$\sum_{n=-\infty}^{\infty} f(x+n) = \sum_{n=-\infty}^{\infty} \hat{f}(n) e^{2\pi i n x}.$$

Note that if $x = 0$,

$$\sum_{n=-\infty}^{\infty} f(n) = \sum_{n=-\infty}^{\infty} \hat{f}(n).$$

We'll apply the Poisson summation formula on a function $f(x)$ with a constant $a > 0$ to be defined as

$$f(x) = \begin{cases} e^{-ax} \text{ if } x > 0 \\ \frac{1}{2} \text{ if } x = 0 \\ 0 \text{ otherwise} \end{cases}.$$

The Fourier transform can easily be computed:

$$\hat{f}(x) = \int_{0}^{\infty} e^{-2\pi i x t} e^{-at}\, dt = \left. \frac{e^{t(-a-2\pi i x)}}{-a-2\pi i x} \right|_{t=0}^{t=\infty} = \frac{1}{a+2\pi i x}.$$

We first calculate the left hand side of the Poisson summation formula:

$$\begin{aligned}\sum_{n=-\infty}^{\infty} f(x+n) &= \frac{1}{2} + \sum_{n=-x+1}^{\infty} e^{-a(x+n)} \\ &= \frac{1}{2} + \frac{e^{-a}}{1-e^{-a}} \\ &= \frac{1+e^{-a}}{2(1-e^{-a})} \\ &= \frac{1}{2}\coth\left(\frac{a}{2}\right).\end{aligned}$$

On the other side, we have

$$\begin{aligned}\sum_{n=-\infty}^{\infty} \hat{f}(n)e^{2\pi inx} &= \sum_{n=-\infty}^{\infty} \frac{1}{a+2\pi in}(\cos(2\pi nx) + i\sin(2\pi nx)) \\ &= \sum_{n=-\infty}^{\infty} \frac{(a\cos(2\pi nx) + i\sin(2\pi nx))(a-2\pi in)}{a^2+(2\pi n)^2} \\ &= \frac{1}{a} + 2\sum_{n=1}^{\infty} \frac{a\cos(2\pi nx) + 2\pi n\sin(2\pi nx)}{a^2+(2\pi n)^2}\end{aligned}$$

Setting $x = 0$,

$$= \frac{1}{a} + 2a\sum_{n=1}^{\infty} \frac{1}{a^2+(2\pi n)^2}.$$

Bringing the two sides together, we have

$$\frac{1}{2}\coth\left(\frac{a}{2}\right) = \frac{1}{a} + 2a\sum_{n=1}^{\infty} \frac{1}{a^2+(2\pi n)^2}.$$

Changing variables by letting $y = \frac{a}{2\pi}$, we have

$$\pi\coth(\pi y) = \frac{1}{y} + 2y\sum_{n=1}^{\infty} \frac{1}{y^2+n^2}.$$

Note that plugging in $a = 1$ leads to the sum

$$\sum_{n=1}^{\infty} \frac{1}{n^2+1} = \frac{1}{2}(\pi\coth(\pi) - 1).$$

Although we don't derive it here, the Poisson Summation Formula can similarly be used to find that

$$\sum_{n=-\infty}^{\infty} \frac{1}{(n+a)^2} = \frac{\pi^2}{\sin^2 \pi a}$$

when $a \in \mathbb{R} \setminus \mathbb{Z}$ [SS11].

References

[Mat] Bessel's inequality and parseval's theorem - page 2.

[Rob99] Neville Robbins. Revisiting an old favorite: zeta(2m). *Mathematics Magazine*, 72(4):317–319, 1999.

[SS11] E.M. Stein and R. Shakarchi. *Fourier Analysis: An Introduction*. Princeton lectures in analysis. Princeton University Press, 2011.