EVALUATING DIVERGENT SERIES

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Abstract. In this paper, we discuss methods for summing divergent series: Cesaro and Abel summation, analytic continuation of the zeta function, Ramanujan summation, and summation under p-adic number systems.

"Divergent series are the invention of the devil, and it is a shame to base on them any demonstration whatsoever." -Niels Henrik Abel

1. Introduction

When discussing infinite series, we usually tend to classify them as convergent or divergent. A series $\sum_{n=0}^{\infty} a_n$ is convergent if $\lim_{N\to\infty} \sum_{n=0}^{N} a_n$ exists and is finite. A series where the limit doesn't exist (because it goes off to infinity, or oscillates forever) is considered divergent. For example, the series $\sum_{n=0}^{\infty} \frac{n}{2^n}$ converges to 2, and $\sum_{n=0}^{\infty} \frac{1}{n}$ and $\sum_{n=0}^{\infty} (-1)^n$ are divergent. Normally, in the study of infinite series, divergent series are simply marked as divergent, without any further investigation. However, there exist meaningful ways of assigning values to divergent series, which we will discuss in this paper.

2. Why divergent series can't be evaluated conventionally

If we try to evaluate a divergent series using conventional means, some awkward issues arise. For example, let's look at the sequence $\sum_{n=0}^{\infty} (-1)^n$. One method of "evaluating" such a series is to pair up the terms: $1-1+1-1+1-1+\cdots = (1-1)+(1-1)+(1-1)+(1-1)+\cdots = 0$. However, if we pair up the terms differently, we get $1+(-1+1)+(-1+1)+... = 1$. Obviously, this is a problem. It turns out that we can't manipulate terms in this way in conditionally convergent series, let alone divergent series. In other cases, if we limit ourselves to the usual methods, the series simply can't be evaluated at all.

Thus, we establish some criteria for methods of evaluating divergent series. Not every method will meet all of these criteria, which may impose some constraints on when they can be used.

Definition 2.1. Linearity: If $\sum a_n = S$, then $\sum ka_n = kS$. Furthermore, if $\sum a_n = S$ and $\sum b_n = T$, then $\sum a_n + b_n = S + T$.

Definition 2.2. Regularity: A method of evaluating divergent series is said to be **regular** or **consistent** if it sums every convergent series to its ordinary sum.

Definition 2.3. Stability: If $\sum_{n=0}^{\infty} a_n = A$, then $\sum_{n=1}^{\infty} a_n = A - a_0$.

3. Cesaro and (C, a) Summation

We refer to [MTG16] and [Har26].

In Cesaro summation, instead of taking the limit of the partial sums, we take the limit of the mean of the partial sums.

Definition 3.1 (Cesaro summation)**.** If we use s_n to denote the n^{th} partial sum then the Cesaro sum of a series is $\sum_{n=0}^{\infty} a_n = \lim_{k\to\infty} \frac{1}{k}\sum_{n=1}^{k} s_n$

Cesaro summation works well for evaluating series where the partial sums oscillate. For example, for the series $\sum_{n=0}^{\infty}(-1)^n$, the first few averages of partial sums are $\frac{1}{1}, \frac{1}{2}, \frac{2}{3}, \frac{1}{2}, \frac{3}{5}$, and so on. Approaching infinity, we have $\lim_{k\to\infty} \frac{1}{k}\sum_{n=0}^{k} s_n = \frac{1}{2}$, which is the Cesaro sum of this series.

However, if the limit of the partial sums goes to infinity, so does the limit of their averages, so such series will not be Cesaro summable.

Later, Cesaro generalized the concept of Cesaro summation to (C, a) summation. The C stands for Cesaro, and a is a nonnegative integer defining the order of this summation. $(C, 0)$ summation is just ordinary summation of a series, and $(C, 1)$ is ordinary Cesaro summation. The higher-degree summation are defined as follows:

Definition 3.2 ((C, a) summation)**.** Given a series $\sum a_n$, define $A_n^{-1} = a_n$ and $A_n^a = \sum_{k=0}^{n} A_k^{a-1}$ (where the upper indices do not denote exponents), and E_n^a to be A_n^a for the series $1 + 0 + 0 + 0 + \cdots$. Then, the (C, a) sum of $\sum a_n$ is defined as $\lim_{n\to\infty} \frac{A_n^a}{E_n^a}$. In fact, (C, a) summation is just the result of iteratively applying Cesaro summation a times.

As the value of a increases, (C, a) summation gets stronger. This means that if a series $\sum a_n$ can be evaluated using (C, k) summation, then it can also be evaluated to the same value using (C, n) summation for any $n > k$.

It can be proven that Cesaro summation is regular, linear, and stable.

4. Abel Summation

Before we discuss Abel summation, we must first introduce Abel's Convergence Theorem.

Theorem 4.1 (Abel's Convergence Theorem)**.** *[Weia] If $G(x) = \sum_{n=0}^{\infty} a_n x^n$ is a power series with real coefficients a_n and radius of convergence R, and $\sum_{n=0}^{\infty} a_n$ converges, then $\lim_{x\to -R^-} G(x) = \sum_{n=0}^{\infty} a_n x^n$. In other words, $G(x)$ is continuous from the left at $x = R$.*

Now, for Abel summation, we apply Abel's Convergence Theorem for $R = 1$.

Definition 4.2. The Abel sum is defined as $\sum_{n=0}^{\infty} a_n = \lim_{z\to -1^-} \sum_{n=0}^{\infty} a_n z^n$.

Let's evaluate the series $\sum_{n=1}^{\infty}(-1)^{n-1}(n)$ using Abel summation. The corresponding power series is

$$f(z) = 1 - 2z + 3z^2 - 4z^3 + ...$$

Integrating gives

$$\int f(z) = C + z - z^2 + z^3 - z^4 + ... = C + \frac{z}{1+z}$$

Then,

$$f(z) = \frac{d}{dz}\int f(z) = \frac{1}{(1+z)^2}$$

Evaluating the left limit gives

$$\sum_{n=1}^{\infty}(-1)^{n-1}(n) = lim_{z\to -1^-} f(z) = \frac{1}{4}$$

Abel summation is also regular, linear, but not always stable.

We see that Abel summation allows the evaluation of some divergent series that can't be evaluated using Cesaro summation, as well as being able to evaluate all Cesaro-summable series. Thus, we say that Abel summation is stronger than Cesaro summation. However, Abel summation still has its limitations; some divergent series, such as $\sum_{n=1}^{\infty} n$, still can't be evaluated.

5. Analytic Continuation of the Zeta Function

Some series that we might be interested in summing, are represented as a function of a variable, like the Riemann zeta function. However, these series generally only converge over part of their domain.

Normally, the zeta function only converges when $Re(s) > 1$. However, we can use analytic continuation to extend the domain of the function to the entire complex plane (except for $s = 1$). Usually, this happens when the function has a power series representation with a radius of convergence extending outside the originally defined domain. Some necessary definitions first:

Definition 5.1 (Analytic Function)**.** [Weic] A function is called analytic on a region R if it is complex differentiable on every point in R.

Definition 5.2 (Analytic Continuation)**.** [Weib] Let f_1 and f_2 be analytic functions on domains Ω_1 and Ω_2, respectively, such that $\Omega_1 \cap \Omega_2$ is not empty and $f_1 = f_2$ on $\Omega_1 \cap \Omega_2$. Then, f_2 is called an analytic continuation of f_1 to Ω_2 and vice versa. If it exists, this analytic continuation is unique.

Definition 5.3 (Zeta function)**.** The Riemann zeta function is a function of a complex variable, defined for $Re(s) > 1$ as follows: $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$. When $Re(s) \leq 1$, then $\zeta(s)$ is defined using analytic continuation.

Definition 5.4 (Bernoulli numbers)**.** [Weid] The Bernoulli numbers are a sequence of rational numbers defined as follows: $B_m = \sum_{k=0}^{m} \sum_{v=0}^{k} (-1)^v \binom{k}{v} \frac{v^m}{k+1}$. Alternatively, they are defined by the exponential generating function $\frac{x}{e^x-1} = \sum_{n=0}^{\infty} \frac{B_n x^n}{n!}$

Theorem 5.5. *[Tao10] $\zeta(-s) = -\frac{B_{s+1}}{s+1}$ after applying analytic continuation to the series definition of the zeta function.*

By plugging in $s = 1$ into Theorem 5.4, we have:

$$\sum_{n=1}^{\infty} n = -\frac{1}{12}$$

For $s = 2$, we have:

$$\sum_{n=1}^{\infty} n^2 = 0$$

In fact, since all the odd-indexed Bernoulli numbers except for B_1 are zero, $\zeta(s)$ is zero when s is an even negative integer. Therefore, using the series definition of the zeta function, for positive even integer k, $\sum_{n=1}^{\infty} n^k = 0$.

This analytic continuation method violates at least one of our criteria. We will show this by counterexample. By definition, analytic continuation is regular. Let's look at what happens when we assume linearity, and add two sums:

$$(A)\sum_{n=1}^{\infty} 1 = 1+1+1+\cdots = -\frac{1}{2}$$

$$(B)\sum_{n=1}^{\infty} n = 1+2+3+\cdots = -\frac{1}{12}$$

If we add them, we end up with

$$(C)\sum_{n=1}^{\infty}(n+1) = 2+3+4+\cdots = -\frac{7}{12}$$

However, if we instead subtract 1 from the first term of (B), we get

$$(D)\sum_{n=2}^{\infty} n = 0+2+3+4+\cdots = -\frac{13}{12}$$

These two equations contradict each other. These series show by counterexample that analytic continuation of the zeta function fails at least one of linearity and stability.

6. Ramanujan Summation

Ramanujan summation extends the Euler-Maclaurin summation formula and expresses the error term using the Bernoulli numbers.

Definition 6.1 (Bernoulli Polynomials)**.** The Bernoulli polynomials are the sequence of polynomials from the generating function $\frac{te^{tx}}{e^t-1} = \sum_{n=0}^{\infty} B_n(x)\frac{t^n}{n!}$.

Definition 6.2 (Periodized Bernoulli Functions)**.** The periodized Bernoulli functions are defined as $P_k(x) = B_k(x - \lfloor x \rfloor)$.

Definition 6.3 (Euler-Maclaurin Formula)**.** If m and n are natural numbers and $f(x)$ is a complex or real valued continuous function for real numbers x in the interval $[m,n]$, then the integral $I = \int_m^n f(x)dx$ can be approximated by the sum $S = f(m+1)+\cdots+f(n-1)+f(n)$. Explicitly, if the function $f(x)$ is p times continuously differentiable, then $S - I = \sum_{k=1}^{p} \frac{B_k}{k!} f^{(k-1)}(n) - f^{(k-1)}(m) + R_p$, where R_p is an error term.

Definition 6.4 (Error Term of the Euler-Maclaurin Formula)**.** The remainder term $R_p = (-1)^{p+1}\int_m^n f^{(p)}(x)\frac{P_p(x)}{p!}dx$

With all the necessary definitions completed, we are now ready to look at Ramanujan summation. We refer to [Ber85]. Ramanujan applied the summation formula for the case $p \to \infty$ to get $\sum_{k=1}^{x} f(k) = C + \int_0^x f(t)dt + \frac{1}{2}f(x) + \sum_{k=1}^{\infty}\frac{B_{2k}}{(2k)!}f^{(2k-1)}(x)$.

In the general case, for functions that converge at $x=1$, we have $C(a) = \int_1^a f(t)dt + \frac{1}{2}f(1) - \sum_{k=1}^{\infty}\frac{B_{2k}}{(2k)!}f^{(2k-1)}(1)$.

For functions $f(x)$, Ramanujan proposed the following definition by letting $x=0$ and $lim_{x\to\infty}R = 0$, then:

Definition 6.5 (Ramanujan Summation)**.** $C(0) = \int_1^0 f(t)dt + \frac{1}{2}f(1) - \sum_{k=1}^{\infty}\frac{B_{2k}}{(2k)!}f^{(2k-1)}(1)$.

It's worth noting that Ramanujan summation is *not* regular. However, it does yield some interesting results: the sum of all the positive integers is $-\frac{1}{12}$, as we found earlier, and the sum of the harmonic series, $\sum_{n=1}^{\Re}\frac{1}{n} = \gamma$, where γ is the Euler-Mascheroni constant.

7. p-adic Number Systems

Instead of assigning a value to a divergent series based on its properties, we can simply use a number system in which the series in question actually converges. For exponential series such as $\sum_{n=0}^{\infty} 2^n$, we can use the p-adic number system to do this.

In our normal decimal number system, we measure the distance between two numbers by the absolute value of their difference. Specifically, $dist(x, y) = |x-y|$. If we did the opposite and used a distance function such that numbers that differed by large prime powers were close together, then

Definition 7.1 (p-adic norms)**.** Let p be a prime. The p-adic norm of 0 is defined to be 0: $|0|_p = 0$. For any nonzero rational x, we write $x = \frac{p^a r}{s}$ for r and s relatively prime to p, and a is maximized. Then, the p-adic norm of x is $|x|_p = p^{-a}$.

We can use this to sum the series $\sum_{n=0}^{\infty} 2^n$. Let's look at the limit of the partial sums: For any positive integer N, $\sum_{n=0}^{N} 2^n = 2^{N+1} - 1$. Now, we'll show that the limit of these partial sums approaches -1. The difference between $2^{N+1} - 1$ and -1 is 2^{N+1}, the norm of which is 2^{-N-1} 2-adically. Thus, as $N \to \infty$, this difference goes to zero 2-adically. When two numbers differ by zero, then they are equal; thus we have $\sum_{n=0}^{\infty} 2^n = -1$.

8. Conclusion

The properties of divergent series lead to several possible ways of assigning them values. While they may not always be consistent with each other, they are still valuable and have many applications in physics or to other areas of math. I hope this paper offered interesting insights into divergent series, a topic not usually covered in discussion of infinite series.

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