COMPLEX ANALYSIS AND INFINITE SERIES

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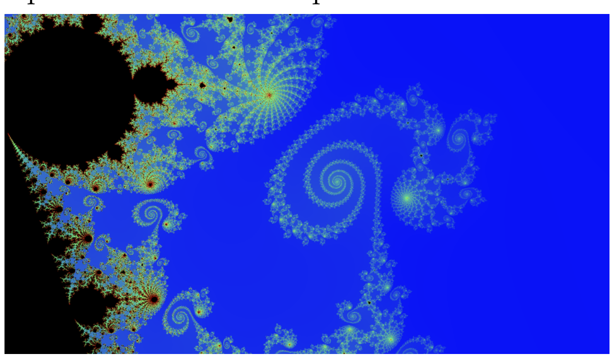
Abstract. In this paper, we will be talking about the basic complex analysis knowledge and some applications in evaluating infinite series with some examples. (Some proofs are shown.)

1. Introduction

Complex analysis, traditionally known as the theory of functions of a complex variable, is the branch of mathematical analysis that investigates functions of complex numbers. It is useful in many branches of mathematics, including algebraic geometry, number theory, analytic combinatorics, applied mathematics; as well as in physics, including the branches of hydrodynamics, thermodynamics, and particularly quantum mechanics. By extension, use of complex analysis also has applications in engineering fields such as nuclear, aerospace, mechanical and electrical engineering.

The world of complex numbers is amazing as the real and the complex parts are interrelated. For example: a sophisticated Julia set picture:



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However, in this paper, we will particularly focus on the employment of complex analysis in evaluating infinite series.

2. complex analysis

2.1. Differentiation.

In this section we define the derivative of a function $f : \mathbf{C} \to \mathbf{C}$ in an analogous manner to the way in which the derivative of a real function is defined, namely as the limit of a difference quotient. To this end we begin with a definition of limit.

Definition 2.1. Let $f : \Omega \to \mathbf{C}$ be defined in a neighbourhood of $z = z_0$. The complex number l is called the limit of f as z approaches z_0 if, given $\epsilon > 0$ there exists $\delta > 0$ such that

$$|f(z) - l| < \epsilon \text{ whenever } 0 < |z - z_0| < \delta.$$

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In this case we write $\lim_{z\to z_0} = l$, Note ϵ and δ are necessarily real. The function f need not be defined at $z = z_0$ in order for this limit to exist. Now we are in a position to define the derivative of a complex function.

Definition 2.2. Let $f : \Omega \to \mathbf{C}$, where Ω is a domain in $\mathbf{C}$. The function f is said to be differentiable at $z \in \Omega$ if the limit

$$\lim_{h\to 0} \frac{f(z+h) - f(z)}{h}.$$

exists. This limit is called the derivative of f at z and is denoted by $f'(z)$. The derivative for complex differentiable functions satifies the same product, quotient, and chain rules as the derivative for real differentiable functions. In particular, it follows that if $f(z) = z^n$, then $f'(z) = nz^{n-1}$. The differentiability of f can be expressed simply in terms of the partial derivatives of the real and imaginary parts of f. Let $z = x + iy$ and $f(z) = u(x, y) + iv(x, y)$, where x, y, u, and v are real. Assuming f is differentiable at $z_0 = x_0 + iy_0$, we will now evaluate the limit along two different paths in the complex plane. First we take the limit along the real axis. If h is restricted to have real values then

$$\frac{f(z+h) - f(z)}{h} = \frac{u(x+h, y) - u(x, y)}{h} + i\frac{v(x+h, y) - v(x, y)}{h},$$

Taking the limit $h \to 0$ gives

$$f'(z) = \frac{\partial u(x, y)}{\partial x} + i\frac{\partial v(x, y)}{\partial x}.$$

Now we will evaluate the limit in (3) along the imaginary axis. To this end, we write $h = ik$ where k is real, which gives

$$\frac{f(z+h) - f(z)}{h} = \frac{v(x, y+k) - v(x, y)}{k} - i\frac{u(x, y+k) - u(x, y)}{k},$$

Taking the limit as $h = ik \to 0$ gives

$$f'(z) = \frac{\partial v(x, y)}{\partial y} - i\frac{\partial u(x, y)}{\partial y},$$

Comparing the real and imaginary parts of the two expressions for $f'(z)$ given in equations above gives the *Cauchy-Riemann equations*

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

Theorem 2.3. *Let $f(z) = u(x, y) + iv(x, y)$ be continuous in some neigbourhood of z. If f is differentiable at z, then u and v satisfy the Cauchy-Riemann equations at z.*

2.2. **Integration.**

Let γ be a curve in the complex plane given by $z(t) = x(t) + iy(t)$, where x and y are smooth real functions of the real variable t in the interval $t_1 < t < t_2$. We define the integral of f along γ by

$$\int_\gamma f(z)dz := \int_{t_1}^{t_2} f(z)\frac{dz}{dt}dt,$$

where the second integral is understood to mean the integral of the real part of the integrand plus i multiplied by the integral of the imaginary part. In terms of line integrals, this becomes

$$\int_\gamma f(z)dz = \int_\gamma \{(udx - vdy) + i(vdx + udy)\}.$$

We will use the symbol " $\oint_\gamma$ " to denote the integral around a closed curve γ. Unless otherwise stated, we will assume that γ is traced in the positive (i.e. anti-clockwise) direction.

Theorem 2.4. *(The Cauchy-Goursat Theorem) Let f be analytic at all points interior to and on a closed curve γ. Then*

$$\oint f(z)dz = 0.$$

Theorem 2.5. *(Cauchy's integral formula) Let f be analytic on the simple closed curve γ and on its interior. Then*

$$f(z) = \frac{1}{2\pi i}\oint_\gamma \frac{f(\zeta)}{\zeta - z}d\zeta$$

Theorem 2.6. *Let f and γ satisfy the conditions of theorem above. Then*

$$f^{(n)}(z) = \frac{n!}{2\pi i}\oint_\gamma \frac{f(\zeta)}{(\zeta - z)^{n+1}}d\zeta.$$

Proof. Assume that equation above is true for some $n = 0, 1, \ldots$ Consider the difference

$$\begin{aligned} f^{(n)}(z+h) - f^{(n)}(z) &= \frac{n!}{2\pi i}\oint_\gamma f(\zeta)\left\{\frac{1}{(\zeta - z - h)^{n+1}} - \frac{1}{(\zeta - z)^{n+1}}\right\}d\zeta \\ &= \frac{n!}{2\pi i}\cdot(n+1)h\oint_\gamma \frac{f(\zeta)}{(\zeta - z)^{n+2}}d\zeta + O\left(h^2\right) \end{aligned}.$$

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2.3. **Taylor Series.** Let f be analytic in the disk $|z - z_0| < r$ for some $r > 0$. Then

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n, \qquad \text{for} \qquad |z - z_0| < r.$$

The series is called the Taylor series for f about $z = z_0$. The Taylor series for f about 0 is called the Maclaurin series of f.

Proof. For fixed $z \in B(z_0, r)$, there is a number r such that $|z - z_0| < \rho < r$. Let γ be the circle with centre z_0 and radius ρ. From Cauchy's integral formula we have

$$f(z) = \frac{1}{2\pi i}\oint_\gamma \frac{f(\zeta)}{\zeta - z}d\zeta.$$

$$\text{Now} \quad \tfrac{1}{\zeta - z} = \tfrac{1}{(\zeta - z_0) - (z - z_0)} = \tfrac{1}{\zeta - z_0}\tfrac{1}{1-w}, \quad w = \tfrac{z - z_0}{\zeta - z_0}\ .$$

Recall that

$$\begin{gathered} \tfrac{1}{1-w} = 1 + w + w^2 + \cdots + w^n + \tfrac{w^{n+1}}{1-w}, \\ sof(z) = \tfrac{1}{2\pi i}\oint_\gamma f(\zeta)\left\{\tfrac{1}{\zeta - z_0} + \tfrac{z - z_0}{(\zeta - z_0)^2} + \cdots + \tfrac{(z-z_0)^n}{(\zeta - z_0)^{n+1}} + \tfrac{1}{\zeta - z}\tfrac{(z - z_0)^{n+1}}{(\zeta - z_0)^n}\right\}d\zeta \end{gathered}.$$

Using theorem 2.5, we have

$$f(z)=f\left(z_0\right)+\frac{f'\left(z_0\right)}{1!}\left(z-z_0\right)+\frac{f''\left(z_0\right)}{2!}\left(z-z_0\right)^2+\cdots+\frac{f^{(n)}\left(z_0\right)}{n!}\left(z-z_0\right)^n+R_n,$$

where

$$R_n=\frac{\left(z-z_0\right)^{n+1}}{2\pi i}\oint_\gamma \frac{f(\zeta)}{(\zeta-z)\left(\zeta-z_0\right)^n}d\zeta.$$

Let $M=\max_{z\in\gamma}|f(z)|$. Then

$$\left|R_n\right| \leq \frac{\left|z-z_0\right|^{n+1}}{2\pi}\cdot 2\pi\cdot\frac{M}{\left(\rho-\left|z-z_0\right|\right)\rho^n}=\frac{M\left|z-z_0\right|}{\rho-\left|z-z_0\right|}\left(\frac{\left|z-z_0\right|}{\rho}\right)^n,$$

since $|z-z_0|<\rho, R_n\to 0$ as $n\to\infty$. ■

2.4. **Isolated Singularities.** A Laurent series is a natural extension of a power series that includes negative powers of the expansion variable. Such series represent functions that are analytic on annuli.

Theorem 2.7. *Any function f that is analytic on the annulus $0\leq r_1<|z-z_0|<r_2\leq\infty$ has a unique Laurent series expansion,*

$$f(z)=\sum_{n=-\infty}^{\infty} a_n\left(z-z_0\right)^n,$$

where

$$a_n=\frac{1}{2\pi i}\oint_\gamma \frac{f(\zeta)}{\left(\zeta-z_0\right)^{n+1}}d\zeta$$

and γ is any circle $|z-z_0|=r$ such that $r_1<r<r_2$. Furthermore the series converges uniformly to $f(z)$ on the annuli.

In the above, the coefficient a_{-1} is called the residue of f at z_0.

Definition 2.8. *A* complex-valued function f is said to have an isolated singularity at $z=z_0$ if there exists $\epsilon>0$ such that f is analytic for all z such that $0<|z-z_0|<\epsilon$ but f is not analytic at $z=z_0$.

Definition 2.9. Let f have an isolated singularity at $z=z_0$ with Laurent series

$$f(z)=\sum_{n=-\infty}^{\infty} a_n\left(z-z_0\right)^n$$

1. If $a_n=0$ for all $n<0$, then f has a removable singularity at $z=z_0$.
2. If there exists a positive integer m such that $a_{-m}\neq 0$ but $a_{-n}=0$ for all $n>m$, then f has a pole of order m at $z=z_0$.
3. If there is no positive integer m such that $a_{-n}=0$ for all $n>m$, then f has an essential singularity at $z=z_0$.

In case 1, the singularity at $z=z_0$ can be removed by extending the definition of f to a function $\tilde{f}$ which is analytic in a neighbourhood of $z=z_0$ given by

$$\tilde{f}(z):=\begin{cases} a_0 & z=z_0\\ f(z) & z\neq z_0\end{cases}.$$

Throughout this module, we will therefore remove any removable singularity and treat it as a regular (i.e. analytic) point.

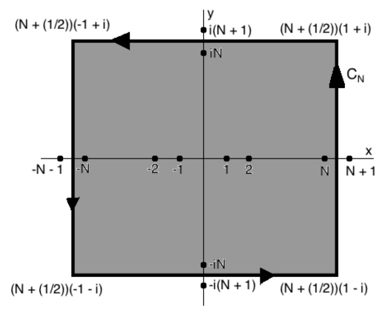
Theorem 2.10. *(The Residue Theorem) Let γ be a closed contour on which a function f is analytic. Let f be analytic on the interior of γ except for a finite number of points $z_1, z_2, \ldots, z_n$ Then*

$$\frac{1}{2\pi i}\oint_\gamma f(z)dz = \sum_{j=1}^{n} r_j,$$

where r_j is the residue of f at $z_j, j = 1, \ldots, n$.

3. Evaluating Infinite Series

We will now develop a general technique to evaluate infinite series of the form $\sum_{n=-\infty}^{\infty} f(n)$ where $f(n)$ is a given function. First let us restrict $f(n)$ to be a meromorphic function (i.e. analytic in $\mathbb{C}$ except for some subset of $\mathbb{C}$), that is f has a finite number of poles, further let f be such that none of these poles are integers. Suppose $G(z)$ is a meromorphic function



whose poles are all simple at $z \in \mathbb{Z}$, and that the residues are all 1. Therefore the residues of $G(z)f(z)$ are $f(n)$. Consider the closed curve C_N, a square that encloses the points $-N, -N+1, \ldots, -1, 0, 1, \ldots, N-1, N$, as seen in figure above. (Note: C_N can be any closed curve enclosing these points).

we know,

$$\oint_{C_N} G(z)f(z)dz = 2\pi i \sum \{\text{ residues of } G(z)f(z) \text{ within } C_N\}.$$

That is to say

$$\oint_{C_N} G(z)f(z)dz = 2\pi i \sum \{\text{ residues of } G(z)f(z) \text{ within } C_N\}$$

$= 2\pi i \sum_{n=-N}^{N} f(n) + 2\pi i \sum \{\text{ residues of } G(z)f(z) \text{ within } C_N \text{ at poles of } f\}$.

So, if $\oint_{C_N} G(z)f(z)dz$ has a convergent limit as C_N gets large, that is as $N \to \infty$, we will be able to conclude things regarding

$$\lim_{N\to\infty} \sum_{n=-N}^{N} f(n) = \sum_{n=-\infty}^{\infty} f(n).$$

Note: If some of f' 's poles are at integers then we can reorder terms such that):

$$\oint_{C_N} G(z)f(z)dz = 2\pi i \sum_{n=-N}^{N} \{f(n) | n \text{ is not a singularity of } f\}$$

$+2\pi i \sum \{\text{ residues of } G(z)f(z) \text{ within } C_N \text{ at poles of } f\}\ \pi\cot(\pi z)$

satisfies the restrictions on $G(z)$ wonderfully, so let $\pi\cot(\pi z) = G(z)$. Following from this we have the summation formula:

$\sum_{n=-\infty}^{\infty}\{f(n)|n$ is not a singularity of $f\} = -\sum\{$ residues of $\pi\cot(\pi z)f(z)$ at singularities of $f\}$, the very tool we wished to develop.

Theorem 3.1. *(Summation Theorem) Let $f(z)$ be analytic in $\mathbb{C}$ except for some finite set of isolated singularities. Also, let $|f(z)| < \frac{M}{|z|^k}$ along the path C_N (shown in figure above), where $k > 1$ and M are constants independent of N. Then we have the summation formula:*

$$\sum_{n=-\infty}^{\infty} f(n) = -\sum \{residues\ of\ \pi\cot(\pi z)f(z)\ at\ f'\ s\ poles\ \}.$$

Example. Prove that $\sum_{n=-\infty}^{\infty}\frac{1}{n^2+a^2} = \frac{\pi}{a}\coth(\pi a)$ where $a > 0$.

Proof. Let $f(z) = \frac{1}{z^2+a^2}$, which has simple poles at $z = \pm ai$ Using Remark 2.2, the residue of $\frac{\pi\cot(\pi z)}{z^2+a^2}$ at $z = ai$ is

$$\lim_{z\to ai}(z-ai)\frac{\pi\cot(\pi z)}{z^2+a^2} = \lim_{z\to ai}(z-ai)\frac{\pi\cot(\pi z)}{(z-ai)(z+ai)} = \frac{\pi\cot(\pi ai)}{2ai} = -\frac{\pi}{2a}\coth(\pi a).$$

Similarly, the residue at $z = -ai$ is $-\frac{\pi}{2a}\coth(\pi a)$ Therefore, the sum of the residues is $-\frac{\pi}{a}\coth(\pi a)$. So, by the Summation Theorem we have

$$\sum_{n=-\infty}^{\infty}\frac{1}{n^2+a^2} = -\left(-\frac{\pi}{2a}\coth(\pi a)\right) = \frac{\pi}{2a}\coth(\pi a).$$

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Example. Prove that $\sum_{n=1}^{\infty}\frac{1}{n^2+a^2} = \frac{\pi}{2a}\coth(\pi a) - \frac{1}{2a^2}$ where $a > 0$.

Proof. Consider the following rewrite of example above, where $\frac{1}{a^2} = f(0)$:

$$\sum_{n=-\infty}^{\infty}\frac{1}{n^2+a^2} = \frac{\pi}{a}\coth(\pi a),$$

$$\sum_{n=-\infty}^{-1}\frac{1}{n^2+a^2} + \frac{1}{a^2} + \sum_{n=1}^{\infty}\frac{1}{n^2+a^2} = \frac{\pi}{a}\coth(\pi a)\ ,$$

$$2\sum_{n=1}^{\infty}\frac{1}{n^2+a^2} + \frac{1}{a^2} = \frac{\pi}{a}\coth(\pi a)$$

since f is even. Therefore, we have

$$\sum_{n=1}^{\infty}\frac{1}{n^2+a^2} = \frac{1}{2}\left(\frac{\pi}{a}\coth(\pi a) - \frac{1}{a^2}\right) = \frac{\pi}{2a}\coth(\pi a) - \frac{1}{2a^2}.$$

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Example. Prove that $\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}$

Proof. Let $f(z) = \frac{1}{z^2} \cdot \cot(z)$ has a simple pole at $z = 0$ because tan (z) has a simple zero there. If the Laurent expansion is $\cot(z) = \frac{b_1}{z} + a_0 + a_1 z + \cdots$, then

$$\left(1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \cdots\right) = \left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots\right)\left(\frac{b_1}{z} + a_0 + a_1 z + \cdots\right).$$

If we multiply, collect terms and then equate coefficients we find that $b_1 = 0, a_0 = 0$ and $a_1 = -\frac{1}{3}$. Thus,

$$\frac{\pi\cot(\pi z)}{z^2} = \frac{\pi\left(\frac{1}{\pi z} - \frac{\pi z}{3} + \cdots\right)}{z^2} = \frac{1}{z^3} - \frac{\pi^2}{3z} + \cdots .$$

Hence the residue of $\frac{\pi\cot(\pi z)}{z^2}$ at $z = 0$ is $\frac{-\pi^2}{3}$.$z = 0$ is the only singularity of f so the summation formula tells us

$$\lim_{N\to\infty} \sum_{n=-N}^{N} \frac{1}{n^2} = \frac{\pi^2}{3},$$

$$\lim_{N\to\infty} \left(\sum_{n=-N}^{-1} \frac{1}{n^2} + \sum_{n=1}^{N} \frac{1}{n^2}\right) = \frac{\pi^2}{3},$$

and because f is even, i.e. $\frac{1}{(-n)^2} = \frac{1}{n^2}$, we see that

$$\lim_{N\to\infty} 2\sum_{n=1}^{N} \frac{1}{n^2} = \frac{\pi^2}{3},$$

$$\lim_{N\to\infty} \sum_{n=1}^{N} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

So we can conclude that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

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References

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[2] Halburd, Rod. "A Very Brief Overview of Complex Analysis." A Very Brief Overview of Complex Analysis, 2009.

[3] SUMMATION OF SERIES USING COMPLEX VARIABLES.