Continued Fractions

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2 Introduction

2.1 Intro

You probably know from school the famous approximation $\pi \approx \frac{22}{7}$, but hidden in that simple fraction is some deep and interesting math of continued fractions. This was studied by Euler himself, and provides deep insights on the real numbers and the nature of irrationality.

3 History

Continued fractions were known to ancient people such as the Greeks and Indians, but they were only used in specific cases, not as an entire field. Several examples were when, in Euclid's Elements, he made an algorithm for finding the gcd, which is highly relevant to continued fractions. Also, Aryabhatiya deals with equations using some continued fractions.

Some notable achievements include when Rafael Bombelli, around the 1500s, expressed $\sqrt{13}$ as a continued fraction.

A major breakthrough came when Brouckner saw the Wallis product for pi ($\frac{3}{2}\frac{3}{4}\frac{5}{4}$...) and was inspired to turn it into the continued fraction, $1+\frac{1^2}{2+\frac{2^2}{3+...}}$. In doing so, he came up with many ideas, such as convergents that are central to continued fractions.

The golden age of continued fractions was when Euler and Lambert realized the connection between continued fractions and irrationality. Euler most famously used this fact to prove that e is irrational.

4 An Example of how to construct a continued fraction

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The best way to grasp continued fractions is to try a simple example. Let's write $\frac{87}{13}$ as a continued fraction. The main way we will do this is to turn improper fractions into proper fractions. $\frac{87}{13}=6+\frac{9}{13}$. You might think that we are done, but what if we flipped $\frac{9}{13}$. Then we could do this again. So $\frac{87}{13}=6+\frac{1}{\frac{13}{9}}$. We can now make $\frac{13}{9}$ proper to get $\frac{87}{13}=6+\frac{1}{1+\frac{4}{9}}$. If we continue, we get $\frac{87}{13}=6+\frac{1}{1+\frac{1}{\frac{9}{4}}}=6+\frac{1}{1+\frac{1}{2+\frac{1}{4}}}$. Now we are actually done because if we flip $\frac{1}{4}$, we will just get 4, which is already proper since it's an integer.

4.1 Three Definitions

1. A **Simple** continued fraction is one where all the numerators are one.

2. let x be represented by the simple continued fraction $a_0+\frac{1}{a_1+\frac{1}{a_2+\frac{1}{a_3+...}}}$. Then we may write $x=[a_0;a_1,a_2,a_3,...]$

3. **Convergent**: If a continued fraction is truncated at the nth term, then it is called the nth convergent of that continued fraction.

4.2 Non-simple Continued Fractions

In our example, we gave $\frac{87}{13}$ a simple continued fraction because all the numerators were 1, but it does not have to be this way. If we allow for integer numerators, we can achieve infinite continued fractions for rational numbers. For instance, $2=\frac{2}{3-2}$. By substituting 2 into 2, we get $2=\frac{2}{3-\frac{2}{3-\frac{2}{3-...}}}$. Here is a warning to be careful with non-simple continued fractions. $1=\frac{2}{3-1}$, so therefore $1=\frac{2}{3-\frac{2}{3-\frac{2}{3-...}}}$ and 2=1.

5 A proof of the bijection between rational numbers and simple finite continued fractions

If you noticed, so far all rational numbers had finite simple continued fractions. Now we shall prove a bijection between them. If we start off with a finite simple continued fraction, then it must be rational, because division and addition by rational numbers is still rational. There is a different proof of this that uses the fact that the extended Euclidean algorithm is strictly decreasing to show that a rational continued fraction must end.

6 Irrational Numbers and Continued Fractions

Continued fractions provide a look at the essence of irrational numbers.

6.1 Approximating irrational numbers with Continued Fractions ($\sqrt{2}, \phi, \pi$)

Let us try to write $\sqrt{2}$ as a continued fraction. To do this we will use difference of squares to "un-rationalize" the denominator. We shall start as $\sqrt{2} = 1 + (\sqrt{2} - 1)$. But by difference of squares, $\sqrt{2} - 1 = \frac{1}{\sqrt{2}+1}$, so we can substitute. $\sqrt{2} = 1 + (\frac{1}{1+\sqrt{2}})$. Now we can do substitution infinitely many times to get $\sqrt{2} = 1 + \frac{1}{2+\frac{1}{2+\frac{1}{2+...}}}$. One way of intuitively checking that we have the correct continued fraction is to take some truncation and see if it approximates the number. For instance, $1 + \frac{1}{2+\frac{1}{2+\frac{1}{2}}} = \frac{17}{12} = 1.416...$, whereas $\sqrt{2} = 1.414....$

Now Let's write the golden ratio as a continued fraction. It is a very famous identity that $\phi = 1+\frac{1}{\phi}$. If we substitute, we get $1 + \frac{1}{1+\frac{1}{1+\frac{1}{1+...}}}$. Now isn't that beautiful?

There are many elegant ways of finding the continued fraction for π(like https://www.ams.org/journals/mcom/ 64-211/S0025-5718-1995-1297479-9/S0025-5718-1995-1297479-9.pdf), but I will focus on a way that requires knowing the decimal approximation. $\pi = 3.141592... = 3 + 0.14592....$ Inverting, we get, $\pi = 3 + \frac{1}{6.85281} = 3 + \frac{1}{6+\frac{1}{1.17259}} = 3 + \frac{1}{6+\frac{1}{1+\frac{1}{5+...}}}$. In fact, if we truncate at $3 + \frac{1}{6+\frac{1}{1}}$, we get the famous approximation $\frac{22}{7}$

6.2 Things To Notice

When we calculated the continued fraction of irrational numbers, you might have noticed 2 things.

1. They have infinite simple continued fractions. This is very important, in fact, there is a bijection between simple infinite continued fractions and irrational numbers.

2. The rational approximations of truncations of continued fractions were good. For instance $\frac{22}{7} \approx 3.14$ is correct to pi by 2 decimal places, and there is no reason to suspect that such a small denominator could approximate a number so well. If we had just gone off the decimal, we would have gotten $\frac{314}{100} = \frac{157}{50}$, which has a much bigger denominator.This is also important because continued fractions provide the $\mathbf{BEST}$ possible rational approximations of irrational numbers. To put it in more formal language, if a rational approximation for α is $\frac{p}{q}$, then for all positive integers m,n such that m<p and n<q, $|\alpha - \frac{p}{q}| < |\alpha - \frac{m}{n}|$.

7 2 Proofs about continued Fractions (best approximation, irrationality

7.1 A proof of Best Approximations

A proof that all best rational approximations of an irrational number come from the truncation of the simple continued fraction of that irrational number. I shall use a rather nice geometric proof from Lorentzen because of its clarity. We shall represent our irrational number α as a line of all points (x,y) such that $\frac{y}{x} = \alpha$. We will represent rational numbers $\frac{p}{q}$ as points (q,p). In this way, rational numbers are just integer lattice points. In this analogy, the best approximation is the lattice point closest to the line. Every subsequent convergent either has a term added to either the numerator or the denominator, odd and even convergent will alternate between above the line and below the line. Thus they form a convex hull of the line and are a best approximation.

7.2 A proof of irrationality

A proof that there is a bijection between infinite simple continued fractions and irrational numbers. In section 3, we proved there was a bijection between rational numbers and simple finite continued fractions.

8 Applications - fast factorization

Not only are continued fractions beautiful, but they have some interesting applications. One such is the continued fraction factorization method (CFRAC), which was the fastest factorization algorithm before Number Field Sieve, and is the basis for both that and Quadratic sieve. It gained fame for factoring $F_1 = 2^{128} + 1$. Let us try to factor the number n. Let the nth convergents of the continued fraction of $\sqrt{n}$ and A_n and B_n. We will define $Q_n = A_n^2 - nB_n^2$, which means that $Q_n \equiv A_n^2 \pmod{n}$

The key insight is that the Q_n are small enough that we can factor some of them such that their product is a square. Then we can have a product of A_ns that equals a different square. We then use Fermat's theorem, which says If $x^2 \equiv y^2 (\mod n)$, then either gcd(x + y, n) or gcd(x - y, n) is a proper factor of n.

Since the Q_n are small, they can easily be factored by trying small primes. Now that we have the factorization of some Q_n, how can we find a product that equals a square? We will use **Gaussian Elimination**.

Gaussian Elimination is a way of reducing matrices efficiently.

We shall Q_n as the nth row of a matrix, and the pth item in the nth row is the power of the pth prime mod 2 in the prime factorization of Q_n.

Once we have reduced the matrix, we can easily add up several rows, to get a square.

Now we are done since, by Fermat's theorem, we have a factor of n.

9 Further study

Because the field of infinite fractions is so expansive, I cannot cover all areas in 1 paper. If you are interested in further study that are somewhat related to infinite fractions and are very interesting, I recommend looking into Rogers–Ramanujan continued fraction, or reading Continued Fractions By: A. Ya. Khinchin as a good book on the subject.

10 Key Terms

Continued fraction: a integer plus a fraction where the denominator may contain another continued fraction

Simple Continued Fraction: a continued fraction where all numerators are 1.

Convergent: A truncation of a continued fraction after n steps

Bijection: A correspondence between to sets where each element in one set correspond to exactly one element in the other set

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