Euler's Pentagonal Number Theorem

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2020

1 Introduction

Euler's Pentagonal Number Theorem was originally proved by Euler in a paper he presented to the St. Petersburg Academy in 1785. As the name implies, it relates to pentagonal numbers, but also to perhaps less obvious mathematical topics, such as the notion of partitions and infinite generating functions. Before delving into the Pentagonal Number Theorem, however, we must first define pentagonal numbers and partitions.

Theorem 1.1. *Pentagonal numbers can be given by the formula* $\frac{n(3n-1)}{2}$.

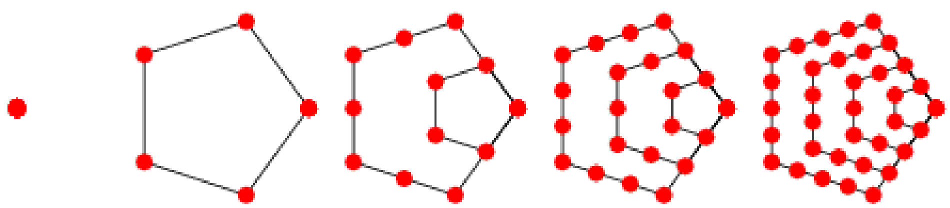

Figure 1

As shown in the diagram above, one can construct a sequence of pentagons with an increasing number of pentagons inside it. Starting at $n = 1$, pentagonal numbers are the name given to the amount of total pentagon vertices in each diagram (represented here by red dots).

Proof. We can visualise how pentagonal numbers are constructed by splitting up the pentagon through the corners as shown in Figure 2.

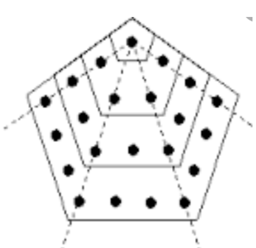

Figure 2

This splits up the lattice pentagon into three triangles, each of which contain $\frac{(n-2)(n-1)}{2}$ lattice points. Adding up the dots located on the separating rays, we get a total of

$$\begin{aligned} &3 \cdot \frac{(n-2)(n-1)}{2} + 4n - 3 \\ &= \frac{3n^2 - 9n + 6 + 8n - 6}{2} \\ &= \frac{n(3n-1)}{2}. \end{aligned}$$

Thus we have the formula for pentagonal numbers. □

Next, we must define what it means to partition an integer n.

Definition 1.1. *Let p_n denote the number of ways one can split up n into a sum of positive integers, where different permutations of the same summands are considered equivalent.*

For example, $5 = 3 + 2$ and $5 = 2 + 3$ are not distinct.

It just so happens that these pentagonal numbers show up in Euler's Pentagonal Number Theorem, which is closely tied to the study of partitions.

Theorem 1.2. *Euler's Pentagonal Number Theorem*

$$\prod_{n=1}^{\infty}(1 - x^n) = \sum_{k=1}^{\infty}(-1)^k x^{k(3k-1)/2} \tag{1}$$

Euler's original proof of (1) requires only the use of algebra, as is presented in Section 2. In Section 3, we will cover alternate proof of (1) as well as an interesting corollary.

2 Euler's Original Proof

First, we define a function $f(x, q)$ that will later become useful as a more general generating function.

Definition 2.1. *Let*

$$f(x, q) = 1 - \sum_{n=1}^{\infty}(1 - xq)(1 - xq^2)(1 - xq^3)...(1 - xq^{n-1})x^{n+1}q^n \tag{2}$$

where we establish the identity:

Lemma 2.1.

$$1 - \sum_{n=1}^{N} = (1 - q)(1 - q^2)(1 - q^3)...(1 - q^{n-1})q^n = \prod_{n=1}^{N}(1 - q^n)$$

Proof. This is a simple proof by induction on N.

Base case: Letting $N = 1$ yields $1 - (1 - 0)q = 1 - q$ as desired.

Inductive step: Assume the lemma holds for N. Then we prove it works for $N + 1$:

$$\begin{aligned}
1 - \sum_{n=1}^{N+1}(1 - q)...(1 - q^{n-1})q^n &= 1 - \sum_{n=1}^{N}(1 - q)(1 - q^2)...(1 - q^{n-1})q^n - (1 - q)(1 - q^2)...(1 - q^N)q^{N+1} \\
&= \prod_{n=1}^{N}(1 - q^n) - (1 - q)(1 - q^2)...(1 - q^N)q^{N+1} \\
&= (1 - q)(1 - q^2)...(1 - q^N) - (1 - q)(1 - q^2)...(1 - q^N)q^{N+1} \\
&= (1 - q)(1 - q^2)...(1 - q^N) \cdot (1 - q^{N+1}) \\
&= \prod_{n=1}^{N+1}(1 - q^n).
\end{aligned}$$

Thus we have completed the proof. □

Now, by extending N to infinity we have a closed formula for $f(1,q)$:

$$f(1,q) = \prod_{n=1}^{\infty}(1-q^n). \tag{3}$$

This is just the right hand side of the Pentagonal Number Theorem! Now we need to find a way to prove that $f(1,q)$ has the alternate representation $\sum_{k=1}^{\infty}(-1)^k x^{k(3k-1)/2}$.

To do this, we'll need a recurrence function for $f(x,q)$. This is because it just so happens that if we iterate the recurrence formula an infinite number of times, we are left with a pleasing closed formula.

Lemma 2.2.

$$f(x,q) = 1 - x^2q - x^3q^2f(xq,q)$$

Proof. We start with Definition 2.1 of $f(x,q)$ and manipulate it:

$$\begin{aligned}
f(x,q) &= 1 - \sum_{n=1}^{\infty}(1-xq)(1-xq^2)(1-xq^3)...(1-xq^{n-1})x^{n+1}q^n \\
&= 1 - x^2q - \sum_{n=2}^{\infty}(1-xq)(1-xq^2)(1-xq^3)...(1-xq^{n-1})x^{n+1}q^n \\
&= 1 - x^2q - \sum_{n=1}^{\infty}(1-xq)(1-xq^2)(1-xq^3)...(1-xq^{n})x^{n+2}q^{n+1} \\
&= 1 - x^2q - \sum_{n=1}^{\infty}(1-xq^2)(1-xq^3)(1-xq^4)...(1-xq^{n})x^{n+2}q^{n+1}(1-xq) \\
&= 1 - x^2q - \sum_{n=1}^{\infty}(1-xq^2)(1-xq^3)(1-xq^4)...(1-xq^{n})x^{n+2}q^{n+1} \\
&\quad + \sum_{n=1}^{\infty}(1-xq^2)(1-xq^3)(1-xq^4)...(1-xq^{n})x^{n+3}q^{n+2} \\
&= 1 - x^2q - x^3q^2 - \sum_{n=2}^{\infty}(1-xq^2)(1-xq^3)(1-xq^4)...(1-xq^{n})x^{n+2}q^{n+1} \\
&\quad + \sum_{n=1}^{\infty}(1-xq^2)(1-xq^3)(1-xq^4)...(1-xq^{n})x^{n+3}q^{n+2} \\
&= 1 - x^2q - x^3q^2 - \sum_{n=1}^{\infty}(1-xq^2)(1-xq^3)(1-xq^4)...(1-xq^{n+1})x^{n+3}q^{n+2} \\
&\quad + \sum_{n=1}^{\infty}(1-xq^2)(1-xq^3)(1-xq^4)...(1-xq^{n})x^{n+3}q^{n+2} \\
&= 1 - x^2q - x^3q^2 - \sum_{n=1}^{\infty}(1-xq^2)(1-xq^3)(1-xq^4)...(1-xq^{n})x^{n+3}q^{n+2}\left[(1-xq^{n+1})-1\right] \\
&= 1 - x^2q - x^3q^2\left(1 - \sum_{n=1}^{\infty}(1-xq^2)(1-xq^3)(1-xq^4)...(1-xq^{n})x^{n+1}q^{2n+1}\right)
\end{aligned}$$

$$= 1 - x^2q - x^3q^2f(xq,q).$$

□

Now the remainder of Euler's proof is just iterating Lemma 2.2 arbitrarily many times:

$$\begin{aligned}
f(x,q) &= 1 - x^2q - x^3q^2f(xq,q) \\
&= 1 - x^2q - x^3q^2\big(1 - x^2q^3 - x^3q^5f(xq^2,q)\big) \\
&= 1 - x^2q - x^3q^2 + x^5q^5 + x^6q^7\big(1 - x^2q^5 - x^3q^8f(xq^3,q)\big) \\
&\quad \cdots
\end{aligned}$$

which always has the closed form

$$\begin{aligned}
&\quad \cdots \\
&= 1 + \sum_{n=1}^{N-1}(-1)^n(x^{3n-1}q^{n(3n-1)/2} + x^{3n}q^{n(3n+1)/2}) \\
&\quad + (-1)^N\left(x^{3N-1}q^{N(3N-1)/2} + x^{3N}q^{N(3N+1)/2}f(xq^N,q)\right)
\end{aligned}$$

But before we move on, we need to verify that we actually get this nice expression!

Lemma 2.3.

$$\begin{aligned}
f(x,q) = 1 &+ \sum_{n=1}^{N-1}(-1)^n(x^{3n-1}q^{n(3n-1)/2} + x^{3n}q^{n(3n+1)/2}) \\
&+ (-1)^Nx^{3N-1}q^{N(3N-1)/2} + (-1)^Nx^{3N}q^{N(3N+1)/2}f(xq^N,q)
\end{aligned}$$

Proof. This is again a proof by induction on N.

Base case: Letting $N = 1$ yields the first equation, $f(x,q) = 1 - x^2q - x^3q^2f(xq,q)$, as desired.

Inductive step: Assume the lemma holds for N. Then we prove that replacing all the Ns with $N+1$'s produces an expression still equal to $f(x,q)$. We do this by iterating Lemma 2.2 one more time:

$$\begin{aligned}
f(x,q) = 1 &+ \sum_{n=1}^{N-1}(-1)^n(x^{3n-1}q^{n(3n-1)/2} + x^{3n}q^{n(3n+1)/2}) \\
&+ (-1)^Nx^{3N-1}q^{N(3N-1)/2} + (-1)^Nx^{3N}q^{N(3N+1)/2}f(xq^N,q) \\
\\
= 1 &+ \sum_{n=1}^{N-1}(-1)^n(x^{3n-1}q^{n(3n-1)/2} + x^{3n}q^{n(3n+1)/2}) \\
&+ (-1)^Nx^{3N-1}q^{N(3N-1)/2} \\
&+ (-1)^Nx^{3N}q^{N(3N+1)/2}(1 - x^2q - x^3q^2f(xq^{N+1},q))
\end{aligned}$$

$$
\begin{aligned}
&= 1 + \sum_{n=1}^{N-1} (-1)^n (x^{3n-1} q^{n(3n-1)/2} + x^{3n} q^{n(3n+1)/2}) \\
&\quad + (-1)^N x^{3N-1} q^{N(3N-1)/2} + (-1)^N x^{3N} q^{N(3N+1)/2} f(xq^N, q) \\
&\quad - (-1)^N x^{3N+2} q^{(3N^2+N+2)/2} - (-1)^N x^{3N+3} q^{(3N^2+N+4)/2} f(xq^{N+1}, q) \\
&= 1 + \sum_{n=1}^{N} (-1)^n (x^{3n-1} q^{n(3n-1)/2} + x^{3n} q^{n(3n+1)/2}) \\
&\quad + (-1)^{N+1} x^{3N+2} q^{(3N^2+N+2)/2} + (-1)^{N+1} x^{3N+3} q^{(3N^2+N+4)/2} f(xq^{N+1}, q)
\end{aligned}
$$

as intended. □

Now, by taking the limit as $N \to \infty$,

$$
f(x, q) = 1 + \sum_{n=1}^{\infty} (-1)^n (x^{3n-1} q^{n(3n-1)/2} + x^{3n} q^{n(3n+1)/2}) \tag{4}
$$

Therefore, by letting $x = 1$ in this equation and putting it together with (2), we complete the Pentagonal Number Theorem:

$$
f(1, q) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n+1)/2} = \prod_{n=1}^{\infty} (1 - q^n)
$$

3 Alternative Proofs

There are several other proofs and corollaries relating to the Pentagonal Number Theorem. In this section we will discuss three such alternate proofs. The first follows from Corollary 3.0.1 and involves the relationship between "even" and "odd" partitions, which will be defined later. The second alternate proof connects infinite products to a generating function for partitions. Finally, the third proof finds pentagonal numbers hidden in a recurrence function for partitions.

The following corollary is a result of plugging in alternate values of x in our function $f(x, q)$.

Corollary 3.0.1. *When multiplied out, the coefficients of each term q^r in the product*

$$
(1 + q)(1 + q^2)(1 + q^3)...(1 + q^{n-1})q^n
$$

is equal to the number of ways one can partition r into distinct summands, where n is the greatest summand.

At first glance one may wonder exactly what this means. Consider $n = 4$. Then our polynomial is

$$
(1 + q)(1 + q^2)(1 + q^3)q^4 = q^4 + q^5 + q^6 + 2q^7 + q^8 + q^9 + q^{10}
$$

This tells us that 6 and 9 can only be split up one way with highest summand 4, (since $6 = 4 + 2$ and $9 = 4 + 3 + 2$) but 7 can be partitioned two ways, namely $4 + 3$ and $4 + 2 + 1$.

We yield this result by letting $x = -1$ in (2) and (4). Thus,

$$f(-1, q) = 1 + \sum_{n=1}^{\infty}(1+q)(1+q^2)\cdots(1+q^{n-1})(-1)^n q^n \tag{5}$$

$$= 1 + \sum_{n=1}^{\infty}(-q^{n(3n-1)/2} + q^{n(3n+1)/2}) \tag{6}$$

In other words, when each term in the sum of (5) is multiplied out,

$$(1+q)(1+q^2)\cdots(1+q^{n-1})q^n$$

produces the polynomial $q^{r_1} + q^{r_2} + q^{r_3} + \cdots + q^{n(n+1)/2}$. Each r exponent is the sum of distinct summands with greatest part n. Therefore, when we take the sum of all the terms of (5), the coefficient of q^r is equal to the number of partitions of r with distinct summands.

Notice the $(-1)^n$ in (5). This determines whether the coefficient will be positive or negative. If the largest summand of q^r is even, the term is positive. Otherwise, the coefficient is negative.

Yet another theorem arises from this observation:

Theorem 3.1. *Let $p_e(n)$ denote the number of ways one can partition n into an even number of parts, and $p_o(n)$ denote the number of ways one can partition n into an odd number of parts. Then:*

$$p_e(n) - p_o(n) = \begin{cases} (-1)^j & \text{if } n = \frac{j(3j\pm1)}{2} \\ 0 & \text{otherwise.} \end{cases}$$

By starting with Theorem 2.3 (obviously using a different proof), it's possible to prove Euler's Pentagonal Number Theorem. We construct the function:

$$1 + \sum_{n=1}^{\infty}(p_e(n) - p_o(n))x^n = \prod_{m=1}^{\infty}(1 - x^m)$$
$$= 1 - x - x^2 + x^5 + x^7 + ...$$

which can also be used to prove the Pentagonal Number Theorem.

It's also possible to start with infinite products of infinite series to get partitions.

Lemma 3.2. $(1 + x + x^2 + ...)(1 + x^2 + x^4 + ...)(1 + x^3 + x^6 + ...)(...) = \sum p_n x^n$*, where p_n denotes the number of partitions of n.*

Intuitively, this makes sense. Every term x^n is created by taking a combination of x^{k_1} from the first sum, x^{2k_2} from the second sum, x^{3k_3} from the third sum, and so on. This multiplies to create $x^n = x^{k_1+2k_2+3k_3+...+sk_s}$. However, we can get more x^ns for all the other ways we can choose $k_1 + 2k_2 + 3_k + ... + sk_s = n$. Thus, the coefficient of x^n is the number of partitions of n.

Theorem 3.3. $\frac{1}{(1-x)(1-x^2)(1-x^3)...} = \sum p_n x^n$

This is Euler's generating function for partitions of an integer. Since x merely plays the role of a placeholder variable, assume $x < 1$. Then the geometric series in Lemma 2.1 converge with ratios x, x^2, x^3, etc.:

$$(1+x+x^2+...)(1+x^2+x^4+...)(1+x^3+x^6+...) = \left(\frac{1}{1-x}\right)\left(\frac{1}{1-x^2}\right)\left(\frac{1}{1-x^3}\right)...$$
$$= \frac{1}{(1-x)(1-x^2)(1-x^3)...} = \sum p_n x^n.$$

The story goes that Euler multiplied out the denominator of this fraction to see if there was a more useful pattern, and that was how he encountered the Pentagonal Number Theorem. Thus we see how partitions can relate to pentagons!

In fact, an alternate proof of (1) starts with partitions. One can define the function $f(n, j)$ as follows.

Definition 3.1. *Let p(n,j) denote the number of ways one can partition n with smallest piece j.*

It follows from this definition that $p(n) = p(n,1)+p(n,2)+p(n,3)+...$ since there is no overlap in the sets that each $p(n, j)$ term counts. We also notice that we can evaluate $p(n, 1)$ by appending 1 to all the sets with sum $n-1$ and smallest term 2. In other words, $p(n, 1) = p(n-1, 1)$. Then $p(n, 2) = p(n-2, 2)$, $p(n, 3) = p(n-3, 3)$, etc. Using the same idea, we get:

$$p(n, j) = \sum_{i=j}^{n} p(n-j, i).$$

Now we can find the recurrence relation for $p(n, j)$. This formula tells us that

$$p(n, j) = p(n-j, j) + p(n-j, j+1) + p(n-j, j+2) + ... \tag{7}$$

but also that

$$p(n-1, j-1) = p(n-j, j-1) + p(n-j, j) + p(n-j, j+1) + ... \tag{8}$$

so subtracting (2) from (1) means:

$$p(n, j) = p(n-1, j-1) - p(n-j, j-1).$$

Eventually we can find a recurrence function for $p(n)$ without the use of $p(n, j)$.

Theorem 3.4. $p(n) = p(n-1) + p(n-2) - p(n-5) - p(n-7) + p(n-12)$ *is the recurrence function when* $p(n)$ *denotes the number of partitions of* n.

Euler multiplied out the denominator of Theorem 2.2 to uncover the pentagonal numbers in the exponents of the expanded polynomial:

$$(1-x)(1-x^2)(1-x^3)... = 1 - x - x^2 + x^5 + x^7 - x^{12} - x^{15} + x^{22} + x^{26} - x^{35} - x^{40} + ...$$

And thus we have gone from partitions to pentagons with very little algebra in between.

References

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